The Nucleolus of Directed Acyclic Graph Games

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Informal definition of the game

i) We have a network that is represented by a directed acyclic graph $G(V, A)$. This graph has a special node - the so called *root* of G, denoted by **r** - such that from each other node of G there leads at least one directed path to the root. This node can be interpreted as a service provider.

ii) Players reside in some of the nodes and they would like to receive the service i.e. get connected to the root.

iii) The arcs of the graph are assigned a non-negative construction cost. For a subgraph T, we define its construction cost $C(T)$ as the total cost of the arcs in T.

iv) For a coalition S, T_S denotes the cheapest subgraph that connect all players in S to the root. The characteristic function value of coalition S is $c(S) = C(T_S)$

This is a well-defined cost game where the question is how to allocate the costs that arise from the construction of the arcs among the players.

Application

Sharing the cost of infrastructural developments, such as building a water pipeline system that connects a group of towns to a water reserve [8].

The old water tower in Landskrona, Sweden

Venn-diagram of graph related cost games

Comparison of graph related cost games

Game	Graph	Edges	Players/node	Convexity	Core
Airport	path	(un)directed	$0 - n$	concave	non-empty
Standard Tree	tree	(un)directed	$0 - n$	concave	non-empty
DAG	connected DAG	directed	$0 - n$	not concave	can be empty
MCST	connected	undirected	1	not concave	non-empty

Complexity

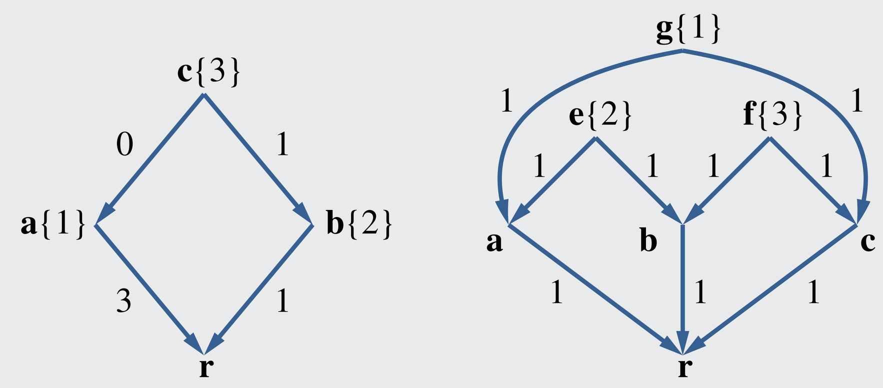
- The game is not concave, thus Kuipers result does not apply [5].
- The problem is closely related to minimum cost spanning tree games, for which finding the nucleolus is NP-hard [1].
- Calculating the characteristic function value for a given coalition, is equivalent with the so-called Steiner arborescence problem, which is NP-hard[4].

This is a hard problem!

Properties of DAG-games

The characteristic function of a DAG-game is

i) non-negative,
ii) monotone,
iii) (strongly) subadditive
iv) but *not* submodular (hence the game is not concave)



Example A Example B

Example A shows a DAG-network for which the associated game is not concave. Let $S_1 = \{1, 3\}$ and $S_2 = \{2, 3\}$, then

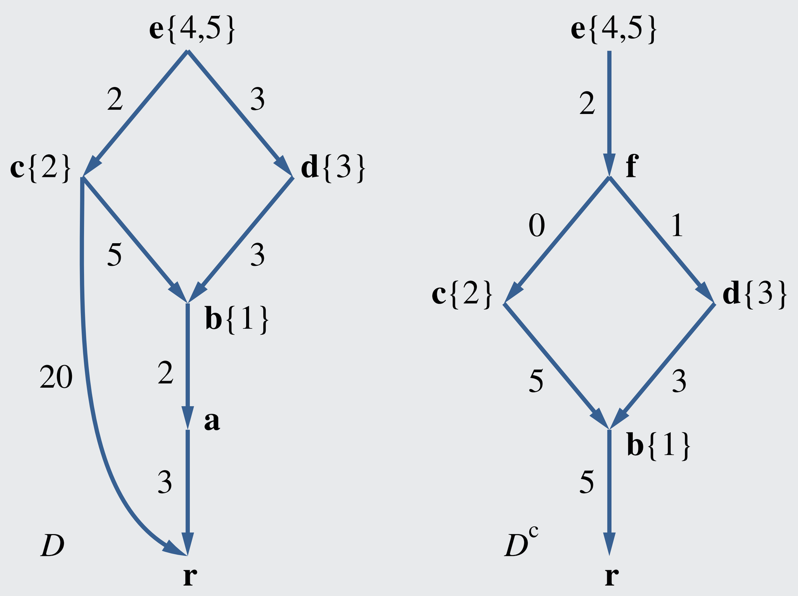
$$3 + 2 = c(S_1) + c(S_2) < c(S_1 \cup S_2) + c(S_1 \cap S_2) = 4 + 2.$$

Example B demonstrates that DAG-games need not even be balanced. The cost of connecting any two-player coalition is 3, however $c(N) = 5$ which leaves the core empty.

Canonization

We say that DAG-game is in canonical form if the following properties are fulfilled:

P1 Each junction has a leaving zero arc.
P2 For each passage the cost of the leaving arc is positive.
P3 There resides a player in each passage.



Sufficient condition for the core

The following condition is sufficient for a DAG-game to have a non-empty core:

(*) There must be a resident at each node with more than one entering arc and with leaving arc(s) all of positive cost.

Notice that this property can be checked efficiently. Furthermore,

i) All networks that satisfy **(*)** can be canonized,
ii) The characteristic function is unaffected by the canonization process.

Construction algorithm for the nucleolus

- The procedure resembles to Maschler, Potters and Reijniers's painting algorithm [7], but the general idea comes from [6], where the lexicographic center of a game is reached by 'pushing hyperplanes' with unit speed.
- The construction algorithm only needs the network as the input. It is not necessary to generate the characteristic function or even a part of it.
- The nucleolus of some classes of DAG-games can be computed in $\mathcal{O}(m^5 \cdot l)$ time, where m denotes the number of nodes and l denotes the number of arcs in the graph. However it seems likely that the running time can be significantly improved with the right implementation.

Structural result

- We characterize 'freeriders' i.e. players who pay nothing in any core allocation.
- We characterize the set of coalitions that have zero excess in the core.
- We identify dually essential coalitions. The core is determined by the efficiency equation $x(N) = c(N)$ and the $x(S) \leq c(S)$ inequalities corresponding to the dually essential coalitions. Moreover this set of coalition also characterize the nucleolus of the DAG-game [3, 2].

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