Constrained multi-issue rationing problems

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Single-issue: rationing problem

A **rationing problem** *is a triple* (N, E, d), *where*

- $N = \{1, 2, \ldots, n\}$ *is a finite set of agents,*
- $E \in \mathbb{R}_{++}$ *is the estate to be distributed among the agents,*
- $d \in \mathbb{R}^N_+$ *is a vector of claims,*
- *We assume that* $E < \sum_{i\in N} d_i$.

A **rationing rule** *is a function* φ, *which associates to each rationing problem* (N, E, d) *a unique point* $\varphi(N, E, d) \in \mathcal{D}(N, E, d)$, *where*

$$\mathcal{D}(N,E,d) = \left\{ x \in \mathbb{R}^N \,\middle|\, \begin{array}{l} \sum_{i\in N} x_i = E \text{ and} \\ 0 \le x_i \le d_i \text{ for all } i \in N \end{array} \right\}$$

Single-issue: Some classical solutions to rationing problems

Constrained equal awards rule, CEA.

$CEA_i(N, E, d) = \min\{\lambda, d_i\}$ for all $i \in N$, where λ satisfies

$$\sum_{i\in N} \min\{\lambda, d_i\} = E.$$

Constrained equal losses rule, CEL.

$CEL_i(N, E, d) = \max\{0, d_i - \lambda\}$ for all $i \in N$, where λ satisfies

$$\sum_{i\in N} \max\{0, d_i - \lambda\} = E.$$

Single-issue: minimal allocation rules

A **minimal allocation rule** MA^α with respect to $\alpha \in \mathbb{R}^N$ assigns to every rationing problem (N, E, d) a vector $MA^\alpha(N, E, d) \in \mathbb{R}^N$ such that

$$MA^\alpha(N,E,d) = \arg\min \left\{ \sum_{i\in N} (x_i - \alpha_i)^2 \,\middle|\, x \in \mathcal{D}(N,E,d) \right\}$$

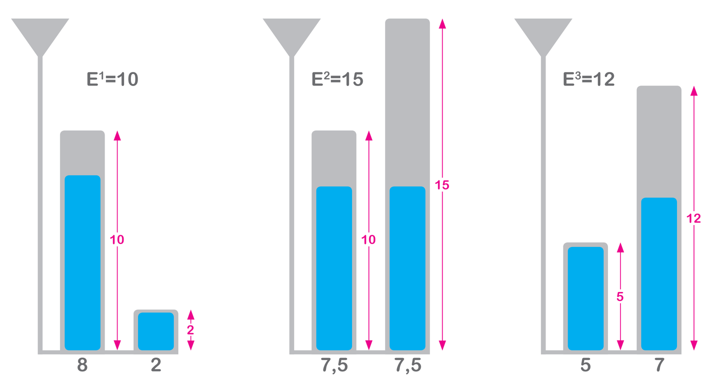
MA^α is well defined since $f(x) = \sum_{i\in N}(x_i - \alpha_i)^2$ is a continuous and strictly convex function and $\mathcal{D}(N, E, d)$ is compact and convex and so the minimization problem has a **unique solution**.

Remark : The parameter α can be interpreted as a reference point.

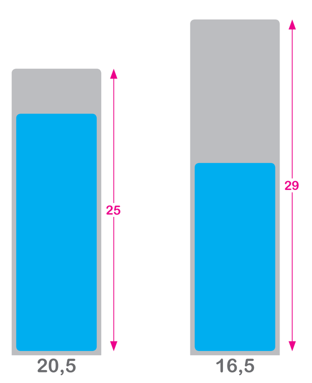
Motivation example: Hydraulic rationing (Kaminski, MSS (2000))

$E^1 = 10$, $E^2 = 15$ and $E^3 = 12$. $d_1 = (d_1^1, d_1^2, d_1^3) = (10, 10, 5)$ and $d_2 = (d_2^1, d_2^2, d_2^3) = (2, 15, 12)$.

Single-issue CEA

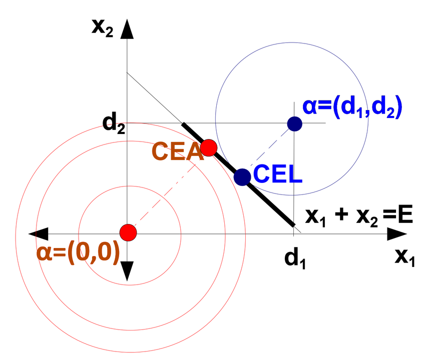


We compare the total payoff of agent 1 and agent 2.



Single-issue: CEA and CEL as a minimal allocation rules

Graphical interpretations of the CEA and the CEL rules for 2 agents as a **minimal allocation rule**:



- If $\alpha = 0 \Rightarrow MA^0 = CEA$.
- If $\alpha = d \Rightarrow MA^d = CEL$.

Single-issue: Generalize rationing problems

A **generalize rationing problem** is a 4-tuple (N, E, d, δ), where N, E and d are defined as be in the *rationing problem* and $\delta \in \mathbb{R}^N_+$ is the vector of endowments of the agents.

For any generalize rationing problem and for all $i \in N$, the **generalized equal awards rule** GEA rule is defined as

$$GEA_i(N, E, d, \delta) = \min\{\max\{0, \lambda - \delta_i\}, d_i\},$$

where λ satisfies $\sum_{i\in N} GEA_i(N, E, d, \delta) = E$.

Schummer, J. and Thomson, W., EL (1997)

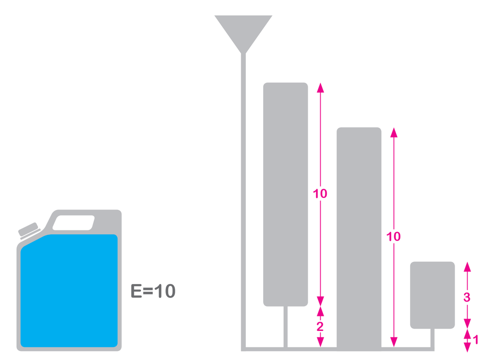
Remark:

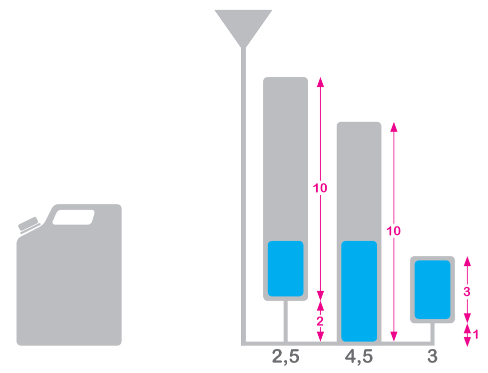
The GEA rule is a generalization of the CEA rule, since $GEA(N, E, d, 0) = CEA(N, E, d)$.

Proposition $GEA(N, E, d, \delta) = MA^{-\delta}(N, E, d)$.

Single-issue: An hydraulic rationing interpretation of the GEA rule

$E = 10$, $d = (10, 10, 5)$ and $\delta = (2, 0, 1)$.





Multi-issue: Constrained multi-issue allocation

Let us suppose that agents claim for different issues $\{1, 2, \ldots, m\}$ and there exist different amounts $E^1, E^2, \ldots, E^m$ corresponding to the different issues that are available to satisfy those claims.

A **constrained multi-issue allocation (CMIA) problem** is a 4-tuple $(N, M, \mathcal{E}, d)$, where

- $N = \{1, 2, \ldots, n\}$ is the set of claimants,
- $M = \{1, 2, \ldots, m\}$ is the set of issues,
- $\mathcal{E} = (E^1, E^2, \ldots, E^m) \in \mathbb{R}^M_{++}$ is the vector of estates,
- $d \in \mathbb{R}^{N\times M}_+$ is the matrix of claims.
- We assume that $E^j < \sum_{i\in N} d_i^j$ for all $j \in M$.

A **CMIA rule** is a function $\hat{\varphi}$, which associates to each CMIA problem $(N, M, \mathcal{E}, d)$ a unique allocation $\hat{\varphi}(N, M, \mathcal{E}, d) \in \mathbb{R}^{N\times M}$ from the following set:

$$\mathcal{D}(N,M,\mathcal{E},d) = \left\{ x \in \mathbb{R}^{N\times M} \,\middle|\, \begin{array}{l} \sum_{i\in N} x_i^j = E^j \text{ for all } j \in M \text{ and} \\ 0 \le x_i^j \le d_i^j \text{ for all } i \in N \text{ and } j \in M \end{array} \right\}$$

Multi-issue: Extended minimal allocation rules

A **extended minimal allocation rule** EMA^α with respect to $(\alpha_i^j)_{\substack{i\in N\\ j\in M}}$ assigns to every CMIA problem $(N, M, \mathcal{E}, d)$ an allocation $EMA^\alpha(N, M, \mathcal{E}, d) \in \mathbb{R}^{N\times M}$ obtained as follows:

Stage 1

$$\mathcal{D}_1 = \arg\min \left\{ \sum_{i\in N} \left(\sum_{j\in M} x_i^j - \sum_{j\in M} \alpha_i^j \right)^2 \,\middle|\, x \in \mathcal{D}(N,M,\mathcal{E},d) \right\}$$

Stage 2

$$EMA^\alpha(N,M,\mathcal{E},d) = \arg\min \left\{ \sum_{i\in N}\sum_{j\in M} \left(x_i^j - \alpha_i^j \right)^2 \,\middle|\, x \in \mathcal{D}_1 \right\}$$

EMA^α is well defined since $\sum_{i\in N}\sum_{j\in M}\left(x_i^j - \alpha_i^j\right)^2$ is a continuous and strictly convex function and $\mathcal{D}_1$ is a compact and convex set and so the minimization problem has a **unique solution**.

Properties of the extended minimal allocation rule

Consistent over agents. For all $T \subseteq N$

$$EMA^\alpha(N,M,\mathcal{E},d)_{|T} = EMA^{\alpha_{|T}}\left(T, M, \left(E^j - \sum_{i\in N\setminus T} EMA_i^{j\alpha}(N,M,\mathcal{E},d)\right)_{j\in M}, d_{|T}\right)$$

- $EMA^\alpha(N, M, \mathcal{E}, d)$ is consistent over agents.

Single-issue consistent. For all $j \in M$

$$EMA^\alpha(N,M,\mathcal{E},d)_{|\{j\}} = EMA^{\alpha'}(N, E^j, d^j),$$

where $\alpha' = \left(\alpha_i^j - \sum_{k\in M\setminus\{j\}} \left(EMA_i^{k\alpha}(N,M,\mathcal{E},d) - \alpha_i^k \right) \right)_{i\in N}$ and $d^j \in \mathbb{R}^N$ is the vector of claims for issue j.

- $EMA^\alpha(N, M, \mathcal{E}, d)$ is single-issue consistent.

Properties of the extended minimal allocation rule

Theorem An extended minimal allocation rule EMA^α is the *ext-CEA* rule if and only if for any $CMIA$ problem $(N, M, \mathcal{E}, d)$ and for all $j \in M$

$$EMA^\alpha(N,M,\mathcal{E},d)_{|\{j\}} = GEA\left(N, E^j, d^j, \left(\delta_i^j = \sum_{k\neq j} x^{*k}_i\right)_{i\in N}\right),$$

where $x^* = EMA^\alpha(N, M, \mathcal{E}, d)$.