Matching with Couples and Bilaterally Substitutable Preferences

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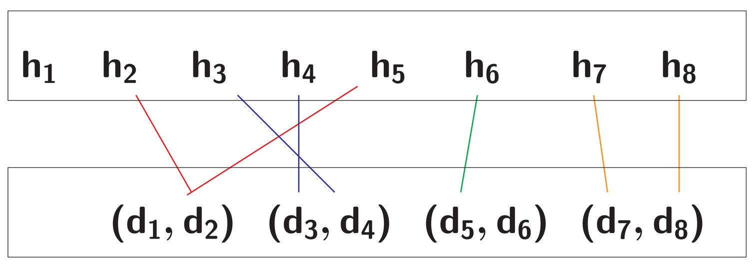
Matching with couples



- One side of the market consists of couples.
- Each couple is matched to (at most) two agents on the other side.

Applications

- Couples
  - National Residency Matching Program (NRMP), allocates single doctors and couples of doctors to hospitals in the U.S.
- Single agents who receive two assignments
  - Scottish resident allocation program (SPA) -Each medical doctor applies for both a medical post and a surgical post.
  - Hungarian higher education matching scheme -Each student applies for a pair of studies.

This Paper

- We look at stability.
- Stability guarantees that there are no agents with the incentive and the power to break the matching.
- Results
  - R.1 A maximal domain for the existence of a stable matching.
  - R.2 Path convergence result, i.e., stability can be reached by means of a decentralized decision making process.

Maximal domain results

Model	Domain	Authors
Matching with couples	Weak responsiveness + unemployment aversion	Klaus & Klijn (2005)
Many-to-many matching with contracts	Substitutability	Hatfield & Kominers (2012)
Matching with couples	Bilateral substitutability	This paper

Path convergence results

Model	Domain/Result	Authors
One-to-one	General preference domain	Roth & Vande Vate (1990)
Roomates	Conditional convergence	Diamantoudi et al. (2004)
Roomates	Convergence to half-stable matchings	Biró et al. (2008)
Many-to-many	Substitutability and responsiveness	Kojima & Ünver (2007)
Couples	Weak responsiveness	Klaus & Klijn (2007)
Couples	Bilateral substitutability	This paper

Model and notation

- Hospitals: $\mathbf{H}$, Couples: $\mathbf{C}$
- The set of Doctors: $\mathbf{D}$ consists of doctors from the couples in $\mathbf{C}$
- There is only one vacant position at each hospital
- For each $\mathbf{h} \in \mathbf{H}$: $\succ_h$ is a strict preference relation over $\mathbf{D}$ and having its position unfilled $\emptyset$
- For each $\mathbf{c} \in \mathbf{C}$: $\succ_c$ is a preference relation over pairs of different hospitals and the outside option $\mathbf{u}$. More precisely, over the set: $\mathcal{H} = [(\mathbf{H} \cup \{\mathbf{u}\}) \times (\mathbf{H} \cup \{\mathbf{u}\})] \setminus \{(\mathbf{h}, \mathbf{h}) : \mathbf{h} \in \mathbf{H}\}$
- For all $\mathcal{H}' \subseteq \mathcal{H}$ and $\mathbf{c} \in \mathbf{C}$, $\mathbf{Ch_c}(\mathcal{H}') := \underset{\succ_c}{\text{argmax}}\{\mathcal{H}'\}$

A matching μ specifies which doctors are matched to which hospitals

One-Sided Blocking Coalitions

Let μ be a matching, $\mathbf{h} \in \mathbf{H}$ and $\mathbf{c} = (\mathbf{d_1}, \mathbf{d_2}) \in \mathbf{C}$

Coalition	One-sided blocking coalition if
$[\mathbf{h}]$	$\emptyset \succ_h \mu(\mathbf{h})$
$[\mathbf{c}, (\mathbf{u}, \mathbf{u})]$	$(\mathbf{u}, \mathbf{u}) \succ_c (\mu(\mathbf{d_1}), \mu(\mathbf{d_2}))$
$[\mathbf{c}, (\mu(\mathbf{d_1}), \mathbf{u})]$	$(\mu(\mathbf{d_1}), \mathbf{u}) \succ_c (\mu(\mathbf{d_1}), \mu(\mathbf{d_2}))$
$[\mathbf{c}, (\mathbf{u}, \mu(\mathbf{d_2}))]$	$(\mathbf{u}, \mu(\mathbf{d_2})) \succ_c (\mu(\mathbf{d_1}), \mu(\mathbf{d_2}))$

Two-sided blocking coalitions

$[\mathbf{c}, (\mathbf{h}, \mathbf{h}')]$, with $(\mathbf{h}, \mathbf{h}') \in \mathcal{H} \setminus \{(\mathbf{u}, \mathbf{u}), (\mu(\mathbf{d_1}), \mathbf{u})), (\mathbf{u}, \mu(\mathbf{d_2}))\}$ is a two-sided blocking coalition for μ if

- $(\mathbf{h}, \mathbf{h}') \succ_c (\mu(\mathbf{d_1}), \mu(\mathbf{d_2}))$
- $[\mathbf{h} \in \mathbf{H}$ implies $\mathbf{d_1} \succsim_h \mu(\mathbf{h})]$ and $[\mathbf{h}' \in \mathbf{H}$ implies $\mathbf{d_2} \succsim_{h'} \mu(\mathbf{h}')]$.

A matching μ is stable if there is no blocking coalition

Bilateral substitutability

For all $\mathcal{H}' \subseteq \mathcal{H}$, and all $\mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3}, \mathbf{h_4} \in \mathbf{H} \cup \{\mathbf{u}\}$ such that $\mathbf{Ch_c}(\mathcal{H}') = (\mathbf{h_1}, \mathbf{h_2})$, $(\mathbf{h_3}, \mathbf{h_4}) \in \mathcal{H}'$, $(\mathbf{h_3}, \mathbf{h_4}) \succsim_c (\mathbf{u}, \mathbf{u})$ and $\{\mathbf{h_1}, \mathbf{h_2}\} \cap \{\mathbf{h_3}, \mathbf{h_4}, \mathbf{u}\} = \emptyset$ we have

$$\big[(\mathbf{bs1})\ (\mathbf{h_1}, \mathbf{h_4}) \succ_c (\mathbf{h_3}, \mathbf{h_4}) \quad \text{or} \quad (\mathbf{bs2})\ (\mathbf{h_1}, \mathbf{u}) \succ_c (\mathbf{h_3}, \mathbf{h_4})\big]$$

and

$$\big[(\mathbf{bs3})\ (\mathbf{h_3}, \mathbf{h_2}) \succ_c (\mathbf{h_3}, \mathbf{h_4}) \quad \text{or} \quad (\mathbf{bs4})\ (\mathbf{u}, \mathbf{h_2}) \succ_c (\mathbf{h_3}, \mathbf{h_4})\big].$$

Theorem (Hatfield & Kojima, 2010) If couples have bilaterally substitutable preferences, then a stable matching exists.

R.1 Maximal Domain Result

If one couple does not have bilaterally substitutable preferences, then it is possible to find preferences for hospitals and bilaterally substitutable preferences for the other couples such that no stable matching exists.

R.2 Path convergence result

Suppose couples have bilaterally substitutable preferences. Let μ be an arbitrary matching. Then, there exists a finite sequence of matchings $\mu_1, \ldots, \mu_k$ such that $\mu_1 = \mu$, μ_k is stable, and for all *i=1,. . . , k-1*, there is a blocking coalition for μ_i such that μ_{i+1} is obtained from μ_i by satisfying this blocking coalition.

Example of a convergent path

Table: A bilaterally substitutable couples market

P^C				P^H				
d_1d_2	d_3d_4	d_5d_6	d_7d_8	h_1	h_2	h_3	h_4	h_5
h_3u	uh_1	h_4h_2	h_2h_3	d_1	d_4	d_1	d_2	d_6
uh_2	h_2h_1	h_1h_2	h_2h_4	d_4	d_3	d_2	d_1	d_8
h_1h_4	h_2u	uh_2	h_2u	d_3	d_1	d_3	d_3	d_7
h_1u	h_5u	h_4u	uh_3	d_2	d_2	d_4	d_8	d_1
uh_4		h_1u	uh_4	d_5	d_7		d_5	d_2
		uh_5			d_8		d_6	d_3

Step					
Initial matching	$\vert d_1d_2$ $\vert uh_4$	d_3d_4 h_5u	d_5d_6 uh_3	d_7d_8 h_2h_1	
(1) Stage 1 eliminate one-sided blocking coalitions	$\vert d_1d_2$ $\vert uh_4$	d_3d_4 h_5u	d_5d_6 uu	d_7d_8 h_2u	h_1, h_3
(2) Stage 2 choose couple (d_1d_2) to enter an initially empty room	d_1d_2 uh_4	$\vert d_3d_4$ $\vert h_5u$	d_5d_6 uu	d_7d_8 h_2u	h_1, h_3
(3) Stage 2 choose couple (d_3d_4) to enter room the room is stable	d_1d_2 uh_4	d_3d_4 h_5u	$\vert d_5d_6$ $\vert uu$	d_7d_8 h_2u	h_1, h_3
(4) Stage 2 choose couple (d_5d_6) to enter room the room is unstable	d_1d_2 uh_4	d_3d_4 h_5u	d_5d_6 uu	$\vert d_7d_8$ $\vert h_2u$	h_1, h_3
(5) Stage 2 satisfy b.c. $[(d_5d_6), (h_5u)]$ the room becomes stable	d_1d_2 uh_4	d_3d_4 uu	d_5d_6 uh_5	$\vert d_7d_8$ $\vert h_2u$	h_1, h_3
(6) Stage 2 choose couple (d_7d_8) to enter room the room is unstable	d_1d_2 uh_4	d_3d_4 uu	d_5d_6 uh_5	d_7d_8 h_2u	$\vert h_1, h_3$
(7) Stage 2 satisfy b.c. $[(d_3d_4), (h_2u)]$ the room becomes stable	d_1d_2 uh_4	d_3d_4 h_2u	d_5d_6 uh_5	d_7d_8 uu	$\vert h_1, h_3$
(8) Stage 3 [h_1 enters the room] satisfy b.c. $[(d_1d_2), (h_1h_4)]$ the room becomes stable	d_1d_2 h_1h_4	d_3d_4 h_2u	d_5d_6 uh_5	d_7d_8 uu	$\vert h_3$
(9) Stage 3 [h_3 enters the room] satisfy b.c. $[(d_1d_2), (h_3u)]$ the room becomes stable	d_1d_2 h_3u	d_3d_4 h_2u	d_5d_6 uh_5	d_7d_8 uu	$\vert h_1, h_4$
(10) Stage 3 [h_1 enters the room] satisfy b.c. $[(d_3d_4), (uh_1)]$ the room becomes stable	d_1d_2 h_3u	d_3d_4 uh_1	d_5d_6 uh_5	d_7d_8 uu	$\vert h_2, h_4$
(11) Stage 3 [h_2 enters the room] satisfy b.c. $[(d_7d_8), (h_2u)]$ the room becomes stable	d_1d_2 h_3u	d_3d_4 uh_1	d_5d_6 uh_5	d_7d_8 h_2u	$\vert h_4$
(12)Stage 3 and Output (h_4 enters). satisfy b.c. $[(d_7d_8), (h_2h_4)]$	d_1d_2 h_3u	d_3d_4 uh_1	d_5d_6 uh_5	d_7d_8 h_2h_4	