The one-seller assignment markets with multiple demands

Francisco Robles and Marina Núñez
University of Barcelona

The assignment game

- *The assignment game is a cooperative model for a two-sided market (Shapley and Shubik, 1972).*
- *Each buyer $i \in M$ demands one unit of an indivisible good and each seller $j' \in M'$ supplies one unit of an indivisible good.*
- *Buyer i and seller j make a join profit of a_{ij} if they trade. This is represented by the assignment matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m'} \\ a_{21} & a_{22} & \dots & a_{2m'} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \cdots & a_{mm'} \end{pmatrix}$$

- *The cooperative game is defined by $(M \cup M', w_A)$, the characteristic function w_A being*

$$w_A(S \cup T) = \max\{ \sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S,T)\},$$

where $S \subseteq M$ and $T \subseteq M'$ and μ is an optimal matching between M and M'.

- *Assignment games have a non-empty core. Moreover:*
- *All buyers maximize their core payoff at the same allocation $(\overline{u}, \underline{v})$, where $(\overline{u}_i = w_A(M \cup M') - w_A((M \setminus \{i\}) \cup M')$.*
- *All sellers maximize their core payoff at the same allocation $(\underline{u}, \overline{v})$, where $\overline{v}_j = w_A(M \cup M') - w_A(M \cup (M' \setminus \{j\}))$*

From Shapley and Shubik (1972) and Gale (1960) we know that for the assignment game, the following sets coincide:

- *The core*
- *The set of solutions of the optimal dual linear assignment problem*
- *The set of pairwise-stable allocations*
- *The set of competitive equilibria payoff vectors*

Multiple partner assignment game

In the multiple-partner assignment model each agent has a quota that determines the maximum number of partnerships he or she can establish. Some related literature is

- *Kaneko (1976).*
- *Sánchez-Soriano* et al. *(2001).*
- *Sotomayor (2002).*
- *Jaume* et al. *(2012).*

Some remarks on multiple-partner assignment game,

- *The core is nonempty in these models, but in general it has no lattice structure*
- *The existence of an optimal core allocation for each side of the market is an open question (in general)*

One-seller assignment markets

Considers a two-sided assignment market with a finite set of buyers and only one seller

- *The seller owns a finite set of indivisible heterogeneous objects on sale*
- *Each buyer $i \in M$ has a capacity or a quota r_i which determines how many objects he can acquire*
- *We introduce many $\sum_{i\in M} r_i$ copies of dummy object, q_0 which are valuated at zero by all buyers. The set of objects owned by the seller is Q_0.*

The one-seller assignment market with multiple demands is $(M, \{0\}, Q_0, A, r)$ where A is the assignment matrix that collects all buyer-object possible gains and $r = (r_1, r_2, ..., r_m) \in \mathbb{Z}_+^M$ specifies the buyers' quotas.

The one-seller assignment game is defined by $(M \cup \{0\}, v_{A,r})$.
The characteristic function is

$$v_{A,r}(S) = \max\left\{ \sum_{(i,j)\in\mu} a_{ij} \,|\mu \in \mathcal{M}(T, Q_0) \right\}$$

for all $S = T \cup \{0\}$ with $\emptyset \neq T \subseteq M$, and $v_{A,r}(S) = 0$ otherwise.

Some results

Theorem: The one-seller assignment game $(M \cup \{0\}, v_{A,r})$ is a Big boss game

- *The core of $(M \cup \{0\}, v_{A,r})$ is nonempty*
- *Nucleolus, τ-value and kernel coincide and pay one-half of his marginal contribution to each buyer*
- *There exists buyers-optimal core allocation $(M_1^{v_A}, M_2^{v_A}, ..., M_m^{v_A}, v_A(M \cup \{0\}) - \sum_{i\in M} M_i^{v_A})$ and seller-optimal core allocation $(0, 0, ..., 0, v_A(M \cup \{0\}))$*

Proposition: The core of the one-seller assignment game $(M \cup \{0\}, v_{A,r})$ is a lattice with a partial order from viewpoint of buyers

Theorem: The one-seller assignment game $(M \cup \{0\}, v_{A,r})$ is convex if and only if for any optimal matching $\mu \in \mathcal{M}_A(M, Q_0)$,

$$\sum_{j\in\mu(i)} a_{ij} \geq \sum_{j\in R} a_{ij},$$

for all $R \subseteq Q_0$ with $|R| \leq r_i$ and for all $i \in M$.

- *Then the Shapley value coincides con the Nucleolus, τ-value and the kernel.*

An Example:

The seller owns $\{1', 2', 3', 0'\}$ the set of buyers is $M = \{1, 2\}$ and their capacities are $r_1 = 2$ and $r_2 = 2$. The assignment matrix is

	1'	2'	3'	0'
1	3	4	3	4
2	10	4	3	4

The characteristic function is,

$$v_A(\{i\}) = 0,\ i = \{1, 2, 0\};\ v_A(\{1, 2\}) = 0;\ v_A(\{1, 0\}) = 14;\ v_A(\{2, 0\}) = 12;\ v_A(\{1, 2, 0\}) = 19$$

Then the core is,

$$C(v_A) = \left\{ (U, V) \in \mathbb{R}_+^2 \times \mathbb{R}_+ \,|U_1 + U_2 + V = 19, U_1 \leq 7, U_2 \leq 5 \right\}$$

Pairwise-stable allocations and Competitive Equilibria

Given a matching $\mu \in \mathcal{M}(M, Q_0)$, a configuration of payoff compatible with μ is $((u_{ij})_{(i,j)\in\mu}, (v_j)_{j\in Q}) = (u, v) \in \mathbb{R}^B \times \mathbb{R}^Q$, where $B = \sum_{i\in M} r_i$, such that:

1. *$u_{ij} + v_j = a_{ij}$, $u_{ij} \geq 0$, $v_j \geq 0$ for all $(i, j) \in \mu$*
2. *$v_j = 0$ if $j \notin \mu(M)$*

A configuration of payoff (u, v) compatible with μ is a pairwise-stable outcome, $((u, v); \mu)$ if $u_{ik} + v_j \geq a_{ij}$ for all $k \in \mu(i)$ and for all $(i, j) \notin \mu$.
Given a pairwise stable outcome, $((u, v); \mu)$, the pairwise-stable payoff vectors for all buyer i is $U_i = \sum_{j\in\mu(i)} u_{ij}$ and for the seller $V = \sum_{j\in Q} v_j$

Given a price vector $p = (p_1, p_2, ..., p_q, 0, ..., 0) \in \mathbb{R}_+^Q$ where p_j is the price of the object j, and every dummy object is priced at zero by the seller, the demand set of buyer $i \in M$ is

$$D_i(p) = \{R \subseteq Q_0, |R| = r_i| \sum_{j\in R} (a_{ij} - p_j) \geq \sum_{j\in R'} (a_{ij} - p_j)\,,$$

for all $R' \subseteq Q_0$, and $|R'| = r_i$

A price p is quasi-competitive if for some $\mu \in \mathcal{M}(M, Q)$, $\mu(i) \in D_i(p)$ for all $i \in M$.

A competitive equilibria is (p, μ) if p is quasi-competitive and for all $j \notin \mu(M)$, $p_j = 0$.

Given a competitive equilibria (p, μ), a competitive equilibria configuration of payoff is

1. *$u_{ij} = a_{ij} - p_j$, for all $(i, j) \in \mu$*
2. *$v_j = p_j$ for all $j \in Q$*

Then, for all buyer i, his payoff is $U_i = \sum_{j\in\mu(i)} (a_{ij} - p_j)$ and for the seller $V = \sum_{j\in Q} p_j$

Proposition: In the one-seller assignment game, the set of competitive equilibria configuration of payoff and the set of pairwise-stable configuration of payoff coincide. (As in the assignment game).

An implementation of the Buyers-optimal core allocation

Given a one-seller assignment game, consider a two-phase mechanism Γ:

1. All buyers play simultaneously. Each buyer $i \in M$ chooses a maximal set of pairs formed by packages and their prices such that these demands make him indifferent.
2. After that, the seller selects a matching μ from buyers to objects such that if $\mu(i) \cap Q \neq \emptyset$ the package $\mu(i)$ belongs to i's demand and pays the price he offered.

The outcome of the mechanism is:

- If a buyer is matched, he receives the package specified by the matching and pays the offered price,
- If a buyer is not matched receives nothing and,
- The seller receives the prices of the matched packages.

Proposition: The buyers-optimal core allocation is attained in a *Subgame Perfect Nash Equilibrium* $(SPNE)$ of the mechanism Γ.

Proposition: Assume that the number of no dummy objects is higher that $\sum_{i\in M} r_i$, then the unique $SPNE$-outcome of the mechanism Γ is the buyers-optimal core allocation.