The Assignment Game with Externalities

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In applications, firms naturally care about which workers are hired by their rivals. We examine the assignment game where profits depend on the matching of the agents.

The Assignment Game without Externalities

There is a set of firms, F, and one of workers, W. Every firm i and worker j jointly create a monetary profit $\alpha_{ij} \geq 0$. A firm hires at most one worker, and no two hire the same.

An **outcome** is a matching μ and payoffs u for the firms and v for the workers such that

$$\sum_{i \in F} u_i + \sum_{j \in W} v_j = \sum_{(i,j) \in \mu} \alpha_{ij}.$$

An outcome is **stable** if it is individually rational and has no blocking pairs:

$$u_i \geq 0, \quad v_j \geq 0, \quad u_i + v_j \geq \alpha_{ij}.$$

A matching μ is **optimal** if

$$\sum_{(i,j) \in \mu} \alpha_{ij} \geq \sum_{(i,j) \in \mu'} \alpha_{ij}$$

for all matchings μ'.

Known Results (Shapley and Shubik, 1971)

For all assignment games without externalities:

- ✓ There exists a stable outcome
- ✓ The stable outcome is optimal
- ✓ The set of stable outcomes coincides with the core
- ✓ The set has a lattice structure: there are worker-optimal and firm-optimal stable outcomes.

Introducing Externalities: An Example

There are three firms and three workers. The following matchings are the only to generate positive profits, listed as (firm, worker):

Matching	Pair 1	Pair 2	Pair 3
$\mu_1 = (1,1), (2,2), (3,3)$	2	2	1
$\mu_2 = (1,2), (2,3), (3,1)$	2	0	2
$\mu_3 = (1,1), (2,3), (3,2)$	2	2	1

Is the following outcome stable?

(μ_1, u, v) such that $u = (1,1,1)$ and $v = (1,1,0)$

When to Block?

Consider firm 2 and worker 3, $u_2 + v_3 = 1$. If they block, it leads to either μ_2 or μ_3. In the former case they fall short, only creating a zero profit; in the latter, they benefit. Agents hence now need to take profits for *all* contingencies into account.

Based on their beliefs about other agents and their attitude towards risk, all pairs calculate thresholds d_{ij} and block outcomes whenever $u_i + v_j < d_{ij}$.

A pair is **reasonable** if

$$p_{ij} = \min_{\mu' \ni (i,j)} \alpha_{ij}^{\mu'} \leq d_{ij} \leq \max_{\mu' \ni (i,j)} \alpha_{ij}^{\mu'} = o_{ij}.$$

A pair is **pessimistic** if $d_{ij} = p_{ij}$: it forms a blocking pair only if it is sure to strictly benefit.

The Assignment Game with Externalities

The monetary profits are matching-specific,

$$\alpha_{ij}^{\mu} \geq 0.$$

An outcome is **stable** if it is individually rational and there are no blocking pairs:

$$u_i \geq 0, \quad v_j \geq 0, \quad u_i + v_j \geq d_{ij}.$$

Which Results Carry Over, and When?

If all pairs are pessimistic:

- ✓ There exists a stable outcome
- ✗ A stable outcome may not be optimal
- ✗ An optimal outcome may not be stable
- ✓ There exists a Pareto efficient stable outcome.

If not all pairs are pessimistic:

- ✗ There may not exist a stable outcome
- ✗ All stable outcomes may be Pareto inefficient.

Ongoing and Future Research

Lattice structure for set of stable outcomes? What if agents use more complex blocking strategies? …

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