Fair Lotteries over Indivisible Objects

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Introduction

When resources are indivisible, such as for example school seats, it is often inevitable that the eventual allocation will seem "unfair" (think two agents, one diamond, one stone). Applying a lottery may then allow us to restore fairness at least from an ex-ante perspective. But when exactly should we call a lottery fair? For this, we draw on the rich literature on fair allocation and adapt various equity criteria to our lottery environment.

Technicalities

We consider the problem of allocating n objects to n agents . Each agent $i \in \mathcal{I}$ is to receive one object $o \in \mathcal{O}$ and holds strict ordinal preferences $\succ_i$ over the set of objects $\mathcal{O}$:

$$o_{1,i} \succ_i o_{2,i} \succ_i \dots \succ_i o_{n,i}.$$

A lottery p assigns object o to agent i with probability p^i_o. The collection $p^i = (p^i_o)_{o \in \mathcal{O}}$ is referred to as the individual lottery of agent i.
To extend agents preferences to (individual) lotteries , we rely on first order stochastic dominance: $p^i \succeq_i \tilde{p}^i$ iff

$$\forall k \in \{1, \dots, n\} : \sum_{l=1}^{k} p^i_{o_{l,i}} \geq \sum_{l=1}^{k} \tilde{p}^i_{o_{l,i}}.$$

Note that first order stochastic dominance only induces a partial ordering over lotteries - this will require us to differentiate between weak and strong versions of prominent equity criteria.

Equity Criteria

Envy-Freeness is arguably the most prominent equity criterion [2]. To check whether an allocation is envy-free, we need to compare agents individual allocations - no agent should then prefer anyone elses lottery.

Definition A lottery p is
- *weakly envy-free* iff $\nexists i \neq j \in \mathcal{I} : p^j \succ_i p^i$.
- *strictly envy-free* iff $\forall i, j \in \mathcal{I} : p^i \succeq_i p^j$.

Prominent example: Probabilistic Serial [1].

Another natural yardstick to measure individuals shares is equal division, here denoted as $(\frac{1}{n})$.

Definition A lottery p satisfies the
- *weak equal division lower bound* iff $\nexists i \in \mathcal{I} : (\frac{1}{n}) \succ_i p^i$.
- *strict equal division lower bound* iff $\forall i \in \mathcal{I} : p^i \succeq_i (\frac{1}{n})$.

Prominent example: Random Serial Dictatorship.

Strict envy-freeness implies strict equal division lower bound; the converse does not hold. The two weak notions are logically independent.

Example 3 agents, preferences $a \succ_i b \succ_i c$.

Then the lottery

	a	b	c
1:	$\frac{11}{20}$	0	$\frac{9}{20}$
2:	$\frac{3}{20}$	$\frac{14}{20}$	$\frac{3}{20}$
3:	$\frac{6}{20}$	$\frac{6}{20}$	$\frac{8}{20}$

is weakly envy-free but fails to meet the weak equal division lower bound.

Example For $i \in \{1, 2\}$ we have $a \succ_i b \succ_i c$ while the third agents most prefered object is c.

Then the lottery

	a	b	c
1:	$\frac{2}{3}$	$\frac{1}{3}$	0
2:	$\frac{1}{3}$	$\frac{2}{3}$	0
3:	0	0	1

meets the strict equal division lower bound but fails to be weakly envy-free as $p^1 \succ_2 p^2$.

In addition, there are various equity criteria for groups of agents, perhaps most notably the *Core from Equal Division* (CfED).

Definition A group of agents $G \subset \mathcal{I}$ may object to a lottery $\tilde{p}$ if there is an alternative lottery p such that
- $\forall o \in \mathcal{O} : \sum_{i \in G} p^i_o = \frac{|G|}{n}$ and
- $\forall i \in G : p^i \succ \tilde{p}^i$.

If there is no such objection that can be raised against a lottery, the lottery is said to be in the core from equal division.

We may also compare the (aggregate) share of one group with the share of other groups. This leads us to the concept of group envy-freeness [4],[5].

Definition. A group of agents $G \subset \mathcal{I}$ may object to a lottery $\tilde{p}$ if there is another group $G' \subset \mathcal{I}$ and a lottery p such that
- $\forall o \in \mathcal{O} : \sum_{i \in G} p^i_o = \frac{|G|}{|G'|} \sum_{j \in G'} \tilde{p}^j_o$ and
- $\forall i \in G : p^i \succ \tilde{p}^i$.

If there is no such objection, the lottery is said to be *group envy-free*. Prominent example: Competitive Equilibrium from Equal Incomes [3]

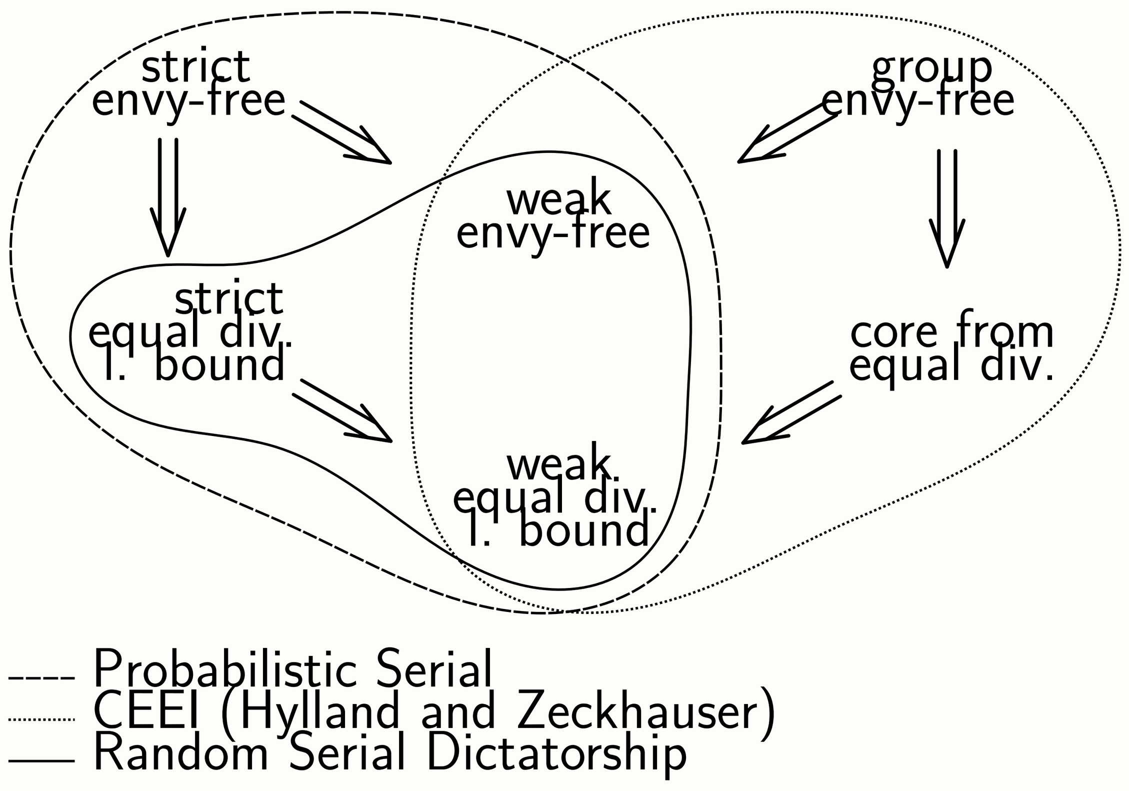

Figure 1 : Logical relations between equity criteria and 3 prominent mechanisms

Main Results

Can we satisfy all criteria simultaneously? No.

Proposition 1 *For all $n \geq 4$, there exist preference profiles $(\succ_i)_{i \in \mathcal{I}}$ such that no lottery simultaneously satisfies strict envy-freeness an lies in the core from equal division.*

Accepting the disconcerting news of Proposition 1, we may ask whether group equity criteria are at least compatible with the strict equal division lower bound. Here the answer is yes.

Proposition 2 *For all n and all preference profiles $(\succ_i)_{i \in \mathcal{I}}$, there exist lotteries in the core from equal endowments that meet the strict equal division lower bound.*

Thus, there is an interesting set of mechanisms between the two contenders Probabilistic Serial and CEEI.

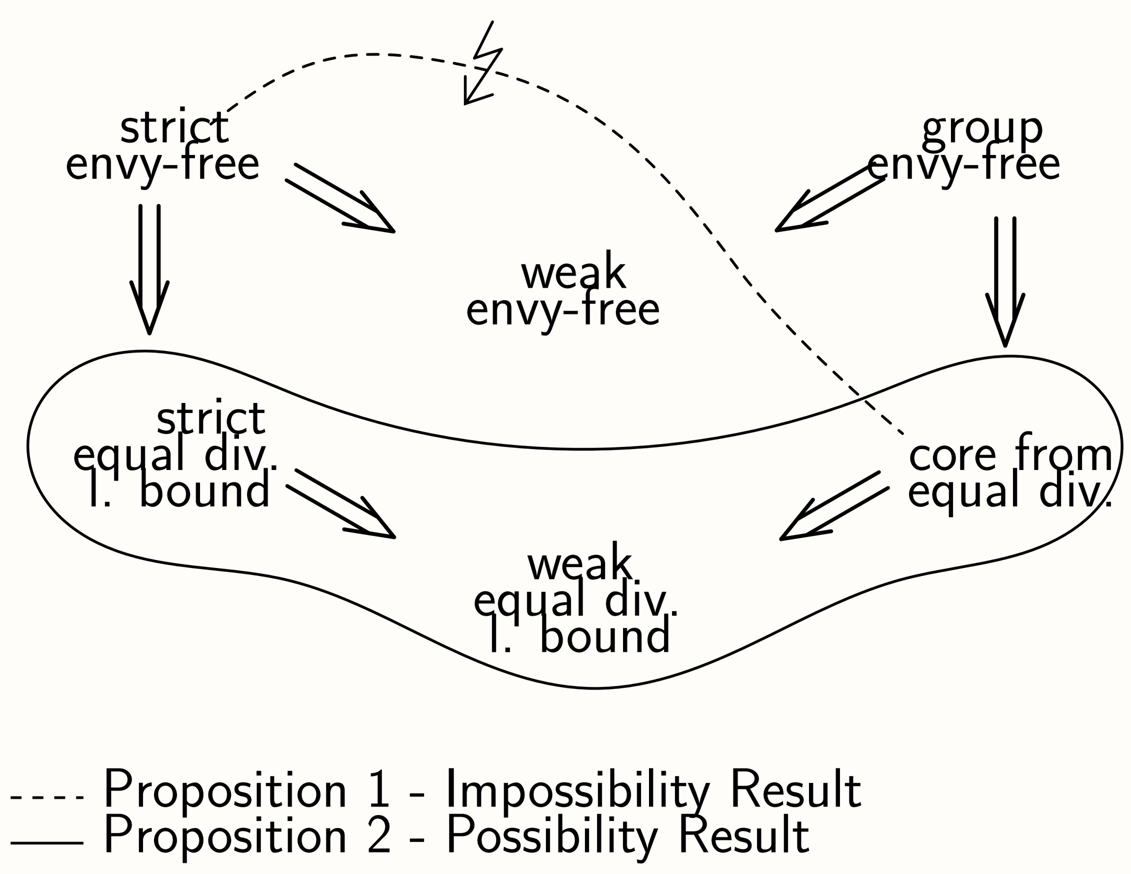

Figure 2 : Main Results

Summary and conclusions

A random allocation mechanism, should be fair, efficient and hard to manipulate. This paper shows that there are many reasonable fairness-criteria to choose from, and that we are indeed forced to choose between them. Moreover, which fairness-criterion we choose to impose, will determine how far we can go in the other two dimension. Thus we better think carefully about what we mean by "fair".

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Summer School on Matching Problems, Markets, and Mechanisms, 24 - 28 June 2013, Budapest, Hungary