An Ordinally-Efficient and Fair Random Solution to the Roommate Problem

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Introduction and Motivation

- No satisfactory solution to the roommate problem
- Little is known about random mechanisms in agent-matching problems

I propose a random mechanism for the roommate problem that is

- ordinally-efficient, individually rational, anonymous
- readily generalizes to two-sided markets
- based on the (general) probabilistic serial mechanism (Bogmolnaia & Moulin 2001; Budish et al. 2013)

Main application:

- Two-pair kidney exchanges with "European" preferences (e.g. Opelz & Döhler 2007)
- Appropriate due to individual rationality and efficiency
- Manipulability concerns are secondary in this setting

Model

- A set $A = \{1, 2, \ldots, n\}$ of agents to be matched in pairs (with some agents possibly left unmatched)
- Each $i \in A$ has a strict preference $\succ_i$ over A
- *Unacceptabilities* ($i \succ_i j$ for some j) are allowed; i.e. i may prefer no match over a match with j
- Partial order $\succ_i^{SD}$ over ΔA induced by 1st-order stochastic dominance
- A *random mechanism* is a function $\mu : \mathcal{P} \to \Delta\mathcal{M}$, where
  - $\mathcal{P}$ - space of all possible preference profiles
  - $\mathcal{M}$ - space of all possible pairwise matchings
- Can represent each $\mu(\{\succ_i\}_{i \in A})$ as a symmetric $n \times n$ prob. share matrix $[p_{ij}]$ with
  - all rows/columns summing to 1 (i.e. bistochastic)
  - $p_{ij} = p_{ji} = \Pr[i \text{ matched with } j]$
  - $p_{ii} = \Pr[i \text{ unmatched}]$
- Given matrix P, P_i denotes i-th row ($\in \Delta A$)

Implementability

- Some symmetric bistochastic matrices not implementable as a lottery over matchings!
- E.g. P below because $\Pr[\exists \text{ unmatched } i] = 1$ in each matching whenever $n = 3$:

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

Implementability Characterization (Cruse 1975): *A symmetric bistochastic matrix $[p_{ij}]$ can be represented as a lottery over deterministic matchings iff*

$$\sum_{i \in C} \sum_{j \in C \setminus \{i\}} p_{ij} \leq 2k$$

for all $k \in \mathbb{N}$ and $C \subseteq A$ with $|C| = 2k + 1$.

Desiderata of the Mechanism

A mechanism is

- *individually rational* (IR) if $i \succ_i j \Rightarrow p_{ij} = 0$
- *weakly strategy-proof* if $\mu_i(\succ_i', \succ_{-i}) \not\succ_i^{SD} \mu_i(\succ_i, \succ_{-i})$
- *[constrained] ordinally-efficient* if $\nexists$ [IR] implementable $Q : Q_i \succsim_i^{SD} \mu_i(\succ)$ for all i (& $\succ_i^{SD}$ for some)
- *anonymous* if permuting agents' names only permutes the rows/columns of the prob. share matrix

Description of the Roommate Probabilistic Serial Mechanism (RPS)

Time runs continuously, $t_0 = 0$. Each $i \in A$ starts with a unit stock of prob. shares. At each t, each $i \in A$ "eats" shares of her most preferred available $j \in A$ with speed 1. That increases p_{ij} and p_{ji}, and decreases stock of both i and j. An agent exits when her stock is depleted. Mechanism ends when all agents have exited.

Agent j is available to $i \neq j$ at time t if

- they don't find each other unacceptable, and
- they haven't exited the mechanism by time t, and
- none of the C-constraints with $i, j \in C$ binds at t.

By symmetry, we also say pair (i, j) is *available*.

The definition of RPS guarantees:

- implementability (satisfies C-constraints)
- anonymity (ex-ante symmetric)
- individual rationality (availability rules)

RPS Example

- Assume $A = \{1, 2, 3, 4\}$, no unacceptabilities, and each agent ranks agents with lower indices higher
- For the first $1/4$ time units, 1 chooses 2 and everyone else chooses 1:

$$RPS_{t=\frac{1}{4}} = \begin{pmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 \end{pmatrix}$$

- Agent 1 exits at $t = 1/4$
- Until $t = 3/8$, 2 chooses 3, and 3&4 choose 2:

$$\begin{pmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/8 \\ 1/4 & 1/4 & 0 & 0 \\ 1/4 & 1/8 & 0 & 0 \end{pmatrix}$$

- Now the C-constraint with $C = \{1, 2, 3\}$ binds!
- 2&3 have to choose 4 since they are unavailable to each other b/c of the C-constraint
- At $t = 7/16$, 2 exits
- The mechanism ends at $t = 21/32$ with

$$RPS_{t=\frac{21}{32}} = \begin{pmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 & 0 \end{pmatrix}$$

Efficiency Properties

Proposition 1: *The RPS mechanism is constrained ordinally-efficient.*

- ...unlike the random serial dictatorship mechanism for the roommate problem!
- Proof idea simplifies related proofs (e.g. Budish et al.)

An Impossibility Result

- However, RPS is not weakly strategy-proof
- ...even without unacceptabilities!
- W/ unacceptabilities, random serial dictatorship fails as well

Still, RPS is the second best mechanism:

Proposition 2: *There does not exist an anonymous, IR, constrained ordinally-efficient, and weak strategy-proof random mechanism for the roommates problem.*

Proof Sketch of the Efficiency Result

- Assume $\forall i : Q_i \succsim_i^{SD} P_i := RPS_i(\succ)$ & $Q \neq P$
- Consider the earliest event (agent exiting or C-constraint binding) making a pair (i, i') unavailable with $P_i \neq Q_i$
- If that is i exiting, by stochastic dominance:

$$\exists j, k \in A : p_{ij} > q_{ij} \geq 0, p_{ik} < q_{ik}, k \succ_i j$$

- If $i = j$, then i was "eating" her own prob. shares when she exited, even though she prefers k
- Thus pair (i, k) must have become unavailable earlier. Contradiction!
- If $i \neq j$, then

$$\exists l \in A : p_{lj} < q_{lj}, l \succ_j i$$

- $p_{ij} > 0 \Rightarrow i$ or j was "eating" the other's prob. shares, even though they both have a more preferred partner
- Hence one of the pairs (i, k) and (l, j) must have become unavailable earlier. Contradiction!
- The analysis in the case of the earliest event being a C-constraint binding is similar

Extensions and Next Steps

- The one-to-one two-sided matching problem is embedded in the roommate problem
- The mechanism also readily generalizes to many-to-many two-sided matching markets
  - Mechanism properties are preserved

Next steps:

- Comparison to Roth-Sönmez-Ünver's "graph-coloring" approach?
- Computational feasibility?
- Indifferences in preferences?
- Comparison to stability-based mechanisms?

References

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