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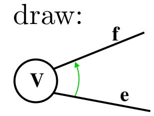
Stable matching

Game Theory in Computer Science
Aarhus Universitet, June 2006

Let G be a graph and for every vertex (agent) v let $<_v$ be a linear order (strict preference) on the edges (connections) incident with v.

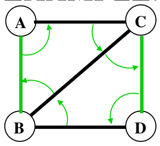
Notation:
f dominates e at v
(denoted by $e <_v f$)

draw:



A matching M is *stable*, if every edge $e \notin M$ is dominated by some edge $f \in M$.

EXAMPLE:



$A : C >_A B$
$B : A >_B C >_B D$
$C : D >_C B >_C A$
$D : B >_D C$

$\{A, C\}, \{B, D\}$ is not stable, since $\{B, C\}$ is a "blocking edge", but $M = \{A, B\} \cup \{C, D\}$ is stable.

"Stable Marriage" if G is bipartite graph

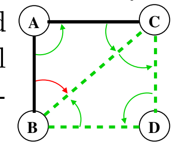
There always exists a stable marriage. (Gale-Shapley, 1962)

Proof: "deferred-acceptance algorithm"
Each man proposes to his most preferred woman and if a woman receives several proposals she accepts the best one and refuses the others... REPEAT

"Stable Roommates" if G is arb. graph

There always exists a stable partition. (Tan, 1991) = *stable half-matching*

There may exist no stable matching:



Here, B, C, D can form a half-weight cycle, and e.g. $\{A, B\}$ is dominated by two half-weight edges.

A P P L I C A T I O N S

Centralized matching programs for two-sided markets:

- **Job-market**
  National Resident Matching Program from 1951, and many others... (see Al Roth's webpage)
- **Student admission**
  Boston Public Schools,
  New York City High Schools,
  Hungarian Universities, etc

Examples for one-sided markets:

- **Chess tournaments**
  (E. Kujansuu et al., 1999)
- **Pairwise kidney exchange**
  (Al Roth, T. Sönmez, U. Ünver, 2004)
- **Firm mergers**
  (N. Angelov, 2006)

P. Biró, K. Cechlárová, T. Fleiner
On the dynamics of stable matching markets

Dynamics

The 17th International Conference on Game Theory at Stony Brook University, July 2006

A new agent enters the market and stability is restored by a "proposal-rejection process"

Two-sided markets:
Roth-Vande Vate (1990)

One-sided markets:
Tan-Hsueh (1995)

[Roth-Sotomayor, 1990] If a woman enters the market and becomes matched, then some men are better off and some women are worse off under ANY stable matching for the new market than at ANY stable matching for the original market.	[Roth-Blum-Rothblum, 1997] Let some men enter the two-sided matching market, then each man either remains matched with the same partner, or receives a worse partner but the best possible in the new market.	[Blum-Rothblum, 2002] In the incremental algorithm if two arrival orders of the agents differs only for one particular agent v, then v gets at least as good partner in the first output, where he arrives later, as in the second, where he arrives earlier.

We have **generalized** the above theorems for one-sided markets.

Key-lemma: If hM_v is a stable half-matching for $G - v$, and edge $\{v, u\}$ is not blocking hM_v, then v and u cannot be matched in a stable half-matching for G.

D.J. Abraham, P. Biró, D.F. Manlove
"Almost Stable" Matchings in the Roommates Problem

Complexity

In Proc. of WAOA 2005: the 3rd Workshop on Approximation and Online Algorithms, LNCS, 3879 , pp 1-14

The problem is to find a matching M s.t.:	where M is	bipartite graph		arb. graph	
		strict pref.	with ties	strict pref.	with ties
M is stable	arb.	Polynomial	Polynomial	Polynomial	NP-hard[1]
	max	Polynomial	NP-hard[2]	Polynomial	(NP-hard)
M has min no. of blocking pairs	arb.	Polynomial	Polynomial	**NP-hard**[3]	**NP-hard**[4]
	max	NP-hard[5]	(NP-hard)	(NP-hard)	(NP-hard)

References:
1: E. Ronn, *J. of Alg.*, 1990 (see also R.W. Irving, D.F. Manlove, *J. of Alg.*, 2002)
2: D.F. Manlove et al., *TCS*, 2002
5: D.F. Manlove, 2006

3: Not approximable within $n^{\frac{1}{2}-\varepsilon}$, 4: Not approximable within $n^{1-\varepsilon}$, (for any $\varepsilon > 0$, unless $P = NP$).