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# Optimal Child-Related Transfers with Endogenous Fertility

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# Optimal Child-Related Transfers with Endogenous Fertility

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Abstract

To compare the systems of child benefits and of family tax deductions, we create a model with endogenous fertility and a basic income, financed from proportional wage taxes. The deduction's efficiency is presumably lower than the benefit's and may even be lower than that of pure basic income.

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JEL classification: J13

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# Társadalmilag optimális transzferek endogén termékenység esetén

Simonovits András

Összefoglaló

A családi pótlék és a családi adókedvezmény rendszerének összehasonlításához egy modellt konstruálunk, amelyben a termékenység endogén és az alapjövedelmet arányos béradó fedezi. A kedvezmény hatékonysága feltehetőleg kisebb, mint a családi pótléké, és még a tiszta alapjövedeleménél is kisebb lehet.

Tárgyszavak: endogén termékenység, transzferrendszerek, személyi jövedelemadó, családi pótlék, családi adókedvezmény

JEL kód: J13

trafer3, direct, no framework

# Optimal Child-Related Transfers with Endogenous Fertility

by András Simonovits

February 27, 2015

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**Abstract***

To compare the systems of child benefits and of family tax deductions, we create a model with endogenous fertility and a basic income, also financed from proportional wage taxes. The deduction's efficiency is presumably lower than the benefit's and may even be lower than that of pure basic income.

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## 1. Introduction

After the baby boom has petered out, below-reproduction fertility rates have become a great problem in a number of developed countries. As a reaction, the concerned governments have been using fertility-related transfer schemes to support families and promote fertility. Note that these schemes vary across time and space. In addition to free school and health care, various financial family support systems exist. To name just the two main types: the *child benefit* is increasing with the number of children, while the *family tax deduction* is a child benefit which only applies up to the income tax obligation. Note that these fertility-related systems operate together with an income-dependent tax system and even interact with the public pension system. In this paper, we will create and analyze a model of optimal fertility-related transfers with endogenous fertility.

To model such transfer systems, the framework of a Stackelberg-game is used: the government announces a transfer rule, and calculating with the transfers, the workers decide on their individual optima. Anticipating these reactions, the government determines the transfer rule by maximizing a social welfare function. (Even if the government does not maximizes any social welfare function, this technique is appropriate for evaluating different transfer systems.) Taking into account individual and social budget constraints calls for general equilibrium analysis.

In models of endogenous fertility (e.g. Becker, 1960; 1991 and Becker and Barro, 1988), when workers decide on their fertility, they consider that more children means less adult consumption but more joy. As is usual, it is heroically assumed a unisex world, where the number of children can be any positive real, including the irrational number $\sqrt{2}$.

Though family support and pension appear as Siamese twins (e.g. Groezen, Leers and Meijdam (2003), for short, GLM theoretically and Gábos, Gál and Kézdi (2009) empirically), in this paper we shall neglect the latter and try to deepen the analysis of the former. To study the properties of a child benefit system, one can neglect income (wage) taxation. But to analyze a family tax deduction system, one needs to model a tax system. And then it is natural to model wage heterogeneity as well. Unfortunately, this dimension has been much neglected in the *theoretical* literature on fertility-related transfers. Making up this omission is the main aim of the present paper. In contrast, the *applied* modelers (e.g. Haan and Wrohlich, 2011) analyzed these complications in much detail but they constructed the social welfare function to fit the real data rather than the other way around.

Deviating from earlier models (e.g. GLM and Simonovits (2013)), we shall assume that the raising costs are proportional to post- rather than pre-tax income or independent of them. This modification greatly complicates the analysis but eliminates possible paradoxes.

To simplify the analysis, either a child benefit system or a family tax deduction system is studied. In our model, the income tax is simply a wage tax which is proportional to the wage, direct redistribution among the adults is achieved via a uniform basic income, while the child benefit is proportional to the number of children. For simplicity, the parent's utility function is an additive logarithmic function of the adult consumption and of fertility.

Our social welfare function is the expected value of the workers' maximal utilities, but á la Feldstein (1985), the paternalistic government typically attributes higher relative weight to the joy of having children than the parents do (represented by preference parameter). Using a generalized utilitarian social welfare function, the redistributive preferences could be studied as well. Since the (pre-tax) wages, the labor supply and the share of reported earning are given, we also fix the tax rate, and look for the socially optimal basic income, benefit rate and tax deduction rate as a function of the tax rate. (Otherwise we could claim the social optimality of full income equality.)

We have only two qualitative analytical result for the child benefit system. Theorem 1: for any fixed tax rate and the paternalistic preference parameter and every wage, the specific fertility rate is an increasing function of the child benefit rate, a decreasing function of the basic income and again an increasing function of the balanced child benefit rate. Moreover, the higher the wage, the weaker are these influences. Theorem 2: low enough benefit is better than no benefit at all. We add, however, two conjectures: within bounds, for a fixed tax rate, 1) the more paternalistic the government or 2) the higher the minimal wage, the higher is the optimal child benefit rate and the lower is the optimal basic income.

Turning to the family tax deduction system and confining the examination to two types, the optimal basic income and the tax deduction rate are independent of the paternalistic preference parameter. Furthermore, when the tax rate rises, only the high-wage workers' fertility rises, and the average fertility is lagging behind the former's rise. Concerning the social welfare, the "optimal" family tax deduction system may yield even a lower welfare than the pure basic income system, at least for progressive social welfare functions. Note that we only support our conjectures by numerical examples.

We call the reader's attention to our model's limitations within the theoretical field: sterility is neglected (for asymmetric information, see Cremer, Gahvari and Pestieau, 2009), the labor supply is fixed (for flexible labor, see Fenge and Meier, 2009) and differences in the relative raising costs and the utility functions, especially in the parameter of relative utility of a child are glossed over (cf. Simonovits, 2013). The role of social norms are also neglected (for its analytical treatment, see e.g. Lindbeck, Nyberg and Weibull, 1999; for a rich empirical discussion of developing countries, see Banerjee and Duflo, 2011, Chapter 5). The activities occurring outside the market are especially important in raising children (e.g. Lee and Mason eds. (2011) and Gál–Szabó–Vargha, 2014) but they are also skipped over. Further research can only clarify how much the message of the paper changes if one takes into account these complexities.

The remainder is organized as follows: Section 2 investigates the special model of child benefits. Section 3 discusses a similar model of family tax deduction and Section 4 concludes. An Appendix shows the pitfalls of two linearizations, either potentially or actually used in the literature (e.g. GLM and Simonovits, 2013).

## 2. Child benefit

There is a unisex population, where adults give birth to children and pay taxes to finance fertility-related transfers and a universal basic income. We assume that all the foregoing functions are well-behaved, i.e. concave and smooth except at certain boundaries.

Every adult in our population is characterized by her pre-tax wage $w$, which is distributed according to a function $F$ and the distribution defines a corresponding expectation operator $\mathbf{E}$. Any adult can freely choose the number of her children, denoted by $n$. There is a transfer system with two objectives: (i) to diminish pre-tax wage inequalities and (ii) to finance a part of the raising costs: a worker earning wage $w$ and having $n$ children, pays tax $\theta w$ and receives basic income $\gamma$ and child benefit $\varphi n$. Denoting the net-of-tax rate by $\hat{\theta} = 1 - \theta$, the net income is equal to

$$y = \hat{\theta} w + \gamma + \varphi n. \tag{1}$$

We assume that the raising cost is proportional to the number of children and the net income. Denoting the proportionality constant by $\pi$, the raising cost is equal to $\pi n y$, therefore the adult consumption is given by

$$c = (1 - \pi n)(z + \varphi n), \quad \text{where} \quad z = \hat{\theta} w + \gamma \quad \text{and} \quad n < \frac{1}{\pi}. \tag{2}$$

To avoid absurd cases, we assume that the child benefit is always nonnegative and not greater than the raising cost: $0 \le \varphi \le \pi z$.

Assume that any adult chooses her fertility to maximize an additive logarithmic utility function

$$U(c, n) = \log c + \zeta \log n, \tag{3}$$

where $\zeta > 0$ is the relative individual utility of having children with respect to that of adult consumption.

Inserting (2) into (3) yields the reduced utility function

$$u(n) = \log(z + \varphi n) + \log(1 - \pi n) + \zeta \log n. \tag{4}$$

We assume that the workers neglect the impact of their decisions on the tax balance described in (9) below.

Equating its derivative to zero provides the optimality condition

$$0 = u'(n) = \frac{\varphi}{z + \varphi n} - \frac{\pi}{1 - \pi n} + \frac{\zeta}{n}. \tag{5}$$

Standard algebraic manipulations imply

**Lemma 1.** *In a transfer system with basic income $\gamma$ and child benefit rate $\varphi$, the optimal fertility of a worker with a narrow income $z = \hat{\theta} w + \gamma$ is the positive root of the quadratic equation*

$$F(n, w, \varphi, \gamma) = (2 + \zeta)\pi\varphi n^2 + \bar{\zeta}(\pi z - \varphi) n - \zeta z = 0, \quad \text{where} \quad \bar{\zeta} = 1 + \zeta, \tag{6}$$

*namely*

$$n(w, \varphi, \gamma) = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \tag{7}$$

*where*

$$A = (2 + \zeta)\pi\varphi, \qquad B = \bar{\zeta}(\pi z - \varphi) \quad \text{and} \quad C = \zeta z. \tag{8}$$

**Remark.** Though formulas (7)–(8) are helpful in the numerical calculations, their analytical use is very limited. In contrast, (5) is very useful, it implies that—neglecting general equilibrium effects, i.e. the dependence of the basic income on the benefit rate in (9) below—the optimal fertility is an increasing function of the benefit rate $\varphi$ and a decreasing function of the wage $w$. We shall return to the general equilibrium at Theorem 1.

Having finished the individual analysis, we consider the whole population with a given wage distribution and assume that $\mathbf{E}w = 1$.

We shall now discuss the balance condition of the tax system providing positive child benefits: on average, the tax is the sum of the child benefit and the basic income, the former being the product of the child benefit rate and the fertility. The balance equation is given by

$$\theta = \varphi \mathbf{E} n(w, \varphi, \gamma) + \gamma. \tag{9}$$

For a fixed $\theta > 0$, one has to substitute (7)–(8) into (9), and solve the resulting implicit equation for $\gamma[\varphi]$. No explicit solution exists except for no child benefit or for homogeneous wages studied below. From now on, we shall write $n[w, \varphi] = n(w, \varphi, \gamma[\varphi]\,)$ and $c[w, \varphi] = c(w, \varphi, \gamma[\varphi]\,)$ in (2) and (9).

We expect that raising the child benefit rate, the basic income decreases and the fertility rises, but we need to prove this conjecture.

Before continuing the general discussion, we present three very simple examples. We start the discussion with the simplest example.

**Example 1.** If there is no child benefit: $\varphi = 0$, then the balance condition is simply $\theta = \gamma$. The introduction of a pure basic income has no fertility effect, because the optimal (unbalanced or balanced) fertility is independent of the wage:

$$n_0 = n(w, 0, \theta) = \frac{\zeta}{\bar{\zeta}\pi}. \tag{7°}$$

Next we continue with another very simple example.

**Example 2.** We have limited the child benefit rate as $\varphi \le \pi z_m$, where $z_m = \hat{\theta} w_m + \gamma$ is the narrow income derived from the minimal wage $w_m$. In case of equality: $\varphi = \pi z_m$, $B_m = 0$, i.e. the fertility belonging to type $w_m$ is simply

$$n_m = \frac{1}{\pi}\sqrt{\frac{\zeta}{\zeta + 2}}. \tag{7-m}$$

It is easy to see that (7-m) is an upper bound on fertility which is quite sharp for the socially optimal child benefit rate if the tax rate is high enough to accommodate $\varphi = \pi z_m$ and the paternalistic preference parameter is high enough.

Our last example is as follows.

**Example 3.** Assume homogeneous wages: $w \equiv 1$ and use notation $\tilde{\theta} = \theta - \gamma$, then $z = 1 - \tilde{\theta}$ and (9) reduces to $n = \tilde{\theta}/\varphi$. Inserting this into the optimum condition (6) yields the inverse of $\gamma[\varphi]$, namely

$$\varphi[\gamma] = \frac{\zeta\pi(1 - \tilde{\theta})\tilde{\theta}}{\tilde{\theta} + \zeta - (2 + \zeta)\pi\tilde{\theta}}, \quad \text{where} \quad \pi < \frac{\tilde{\theta} + \zeta}{(2 + \zeta)\tilde{\theta}}.$$

In this degenerate example, we have determined an explicit but quite cumbersome trade-off between the child benefit rate $\varphi$ and the basic income $\gamma$.

Returning to the general model with heterogeneous wages and a fixed tax rate $\theta$, we shall prove

**Theorem 1.** *For every wage, the specific optimal fertility is an increasing function of the child benefit rate $\varphi \geq 0$, a decreasing function of the basic income $\gamma \geq 0$ and an increasing function of the balanced child benefit rate on the appropriate interval $[0, \varphi_M]$ defined as*

$$\theta = \varphi_M \mathbf{E} n(w, \varphi_M, 0). \qquad (9-M)$$

**Remark.** The higher the wage, the weaker is the impact of the rise in the transfer rates.

**Proof.** a) Consider the optimality condition (5) and increase $\varphi$, then the declining curve shifts to the right, leading to a higher $n(w, \varphi, \gamma)$. b) The opposite holds for $\gamma$.

c) To prove the third proposition, one needs calculate. Substituting $n(w, \varphi, \gamma[\varphi])$ into (9) and taking its total derivative with respect to $\varphi$ results in

$$0 = \mathbf{E} n(w, \varphi, \gamma[\varphi]) + \varphi \mathbf{E} n'_\varphi(w, \varphi, \gamma[\varphi]) + \varphi \mathbf{E} n'_\gamma(w, \varphi, \gamma[\varphi])\gamma'[\varphi] + \gamma'[\varphi]. \qquad (11)$$

Because $n'_\varphi > 0 > n'_\gamma$, (11) implies $\gamma'[\varphi] < 0$. The balanced marginal change in the fertility is equal to

$$\frac{d}{d\varphi} n(w, \varphi, \gamma[\varphi]) = n'_\varphi + n'_\gamma \gamma'[\varphi] > 0.$$

d) Since the right hand side of (9) is monotone increasing in $\varphi$, $0 = \gamma[\varphi_M]$ has a unique solution etc. ∎

To choose the socially optimal transfer system, the government maximizes a pure utilitarian social welfare function

$$V^o[\varphi] = \mathbf{E}[\log c[w, \varphi] + \zeta \log n[w, \varphi]\,] \qquad (10^o)$$

or its paternalistic version (cf. Feldstein, 1985):

$$V[\varphi] = \mathbf{E}[\log c[w, \varphi] + \zeta^* \log n[w, \varphi]\,], \qquad (10)$$

where $\zeta^* \geq \zeta$ is the paternalistic preference parameter. We shall prove that some positive child benefit is socially useful.

**Theorem 2.** *If $\zeta^* > \zeta$, then socially optimal balanced child benefit rate is positive.*

**Proof.** The basic idea is borrowed from the well-know proof of the envelope-theorem. We shall show $V'[0] > 0$. Taking the derivative of $V$ in (10) with respect to $\varphi$ yields

$$V'[\varphi] = \mathbf{E} \frac{z'_\varphi[w, \varphi] + n[w, \varphi] + \varphi n'_\varphi[w, \varphi]}{z[w, \varphi] + \varphi n[w, \varphi]} + \mathbf{E} \frac{-\pi n'_\varphi[w, \varphi]}{1 - \pi n[w, \varphi]} + \zeta^* \mathbf{E} \frac{n'_\varphi[w, \varphi]}{n[w, \varphi]}.$$

Using the individual optimality condition (5), multiplying it by $n'_\varphi[w,\varphi]$ and using $z'_\varphi[w,\varphi] = \gamma'[\varphi]$, we obtain

$$V'[\varphi] = \mathbf{E}\frac{\gamma'[\varphi] + n[w,\varphi]}{z[w,\varphi] + \varphi n[w,\varphi]} + (\zeta^* - \zeta)\mathbf{E}\frac{n'_\varphi[w,\varphi]}{n[w,\varphi]}.$$

We want to demonstrate the fraction in the first expectation of $V'[0]$ is zero. By (7°) and (11), its numerator is $\gamma'[0] + n[w,0] = n[w,0] - \mathbf{E}n[w,0] = 0$. The second fraction is positive, therefore $V'[0] > 0$. ∎

As is usual, to compare the two systems—basic income combined with child benefits and pure basic income—from a welfare point of view, we introduce the following concept: the *relative efficiency* $\varepsilon$ of the transfer system with respect to the pure basic income is equal to that positive real number, multiplying the wages of the no-benefit system by it, the welfare is equal to that of the benefit system with the original wages. Adding an argument for the average wage $\varepsilon$ in the social welfare function, the corresponding equation for efficiency is

$$V(1,\varphi) = V(\varepsilon,0).$$

Due to the special structure of the utility and the social welfare functions,

$$V(1,\varphi) = V(1,0) + \log\varepsilon, \quad \text{i.e.} \quad \varepsilon = \exp(V(1,\varphi) - V(1,0)).$$

To help the understanding of the system's behavior, we shall display numerically the dependence of the optimal outcomes on the balanced transfer rates. We have two types: $w_L = 0.5$ and $w_H = 2$, with a common relative raising cost $\pi = 0.35$ and population shares $f_L = 2/3$, $f_H = 1 - f_L = 1/3$. We choose three values for the tax rate: $\theta = 0.1$, 0.2 and 0.3 and three values for the paternalistic preference parameter: $\zeta^*_N = 0.4$ (no paternalism), $\zeta^*_W = 0.5$ (weaker paternalism) and $\zeta^*_S = 0.6$ (stronger paternalism). Since the social welfare functions are very flat around the optima, we tabulate their values at steps 0.05 in the benefit rate. The approximate maxima are italicized.

As has been already mentioned above, for $\varphi = 0$, $\gamma = \theta$ and $n_W = n_H = n_0$. For $\theta = 0.1$ (first block), the gap between low- and high fertilities grows together with the balanced benefit rate. The efficiencies $\varepsilon_N$ and $\varepsilon_N$ are almost invariant to the transfer rates, while $\varepsilon_S$ is rising to 1.03. For $\theta = 0.2$ (second block), the maximal fertility 1.166 of (7°) is approximated at $\varphi = 0.15$, namely 1.161. The inefficiency of the no paternalism is visible, the efficiency of the weaker paternalism is weak, and that of the stranger is strong. Finally, for $\theta = 0.3$ (third block), the picture is sharper: the weaker paternalism's optimum lies around $\varphi = 0.1$ and the stronger paternalism's optimum lies around $\varphi = 0.15$, close to the maximum.

**Table 1.** *Impact of the child benefit rate for various θs and ζ*s*

| Tax rate $\theta$ | Child benefit r. $\varphi$ | Basic income $\gamma$ | Fertility: Low wage $n_L$ | Fertility: High wage $n_H$ | Average $n$ | Relative efficiency: No paternalism $\varepsilon_N$ | Relative efficiency: Weak paternalism $\varepsilon_W$ | Relative efficiency: Strong paternalism $\varepsilon_S$ |
|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.00 | 0.100 | 0.816 | 0.816 | 0.816 | *1* | 1 | 1 |
| | 0.05 | 0.055 | 0.933 | 0.848 | 0.904 | 0.998 | 1.008 | 1.018 |
| | 0.10 | 0.000 | 1.061 | 0.882 | 1.001 | 0.990 | *1.010* | *1.030* |
| 0.2 | 0.00 | 0.200 | 0.816 | 0.816 | 0.816 | *1* | 1 | 1 |
| | 0.05 | 0.155 | 0.922 | 0.850 | 0.898 | 0.998 | 1.007 | 1.017 |
| | 0.10 | 0.102 | 1.039 | 0.886 | 0.988 | 0.991 | *1.010* | 1.029 |
| | 0.15 | 0.038 | 1.161 | 0.924 | 1.082 | 0.978 | 1.006 | *1.034* |
| 0.3 | 0.00 | 0.300 | 0.816 | 0.816 | 0.816 | *1* | 1 | 1 |
| | 0.05 | 0.256 | 0.914 | 0.852 | 0.893 | 0.999 | 1.008 | 1.017 |
| | 0.10 | 0.203 | 1.021 | 0.890 | 0.977 | 0.992 | *1.010* | 1.028 |
| | 0.15 | 0.141 | 1.133 | 0.931 | 1.066 | 0.981 | 1.007 | *1.034* |

**Remark.** $\zeta_N^* = 0.4$, $\zeta_W^* = 0.5$ and $\zeta_S^* = 0.6$.

We continue with two conjectures for any fixed tax rate $\theta$. We assume that the optimal benefit rate has not reached its maximum, namely $\pi z_m$ and the corresponding basic income is positive.

**Conjecture 1.** *The socially optimal child benefit rate is an increasing function of the paternalistic preference parameter.*

**Conjecture 2.** *For a fixed paternalistic preference parameter $\zeta^* > \zeta$, the socially optimal child benefit rate is an increasing function of the minimal wage $w_m$.*

Anticipating the discussion of family tax deduction in Section 4, we introduce a further important variable: the excess of child benefit over the tax, for short, *excess transfer*, namely $e = \varphi n - \theta w$. If it is positive, then the type's tax falls short of her child benefit. This is not to be confused with the transfer $\gamma - \theta w$ arising in a pure basic income system which can also be negative.

## 3. Family tax deduction

There are governments which are worried by the large transfers flowing from high-earner workers (families) to low-earner ones through child benefits, especially at pronatalistic policies. To mitigate this unwanted consequence, these governments replace child benefits by family tax deductions. (It is possible to model a partial replacement but it would unnecessarily complicate the analysis.) The essence of the family tax deduction is that only the higher wage types can fully use it: any positive excess transfer $e = \varphi n(w) - \theta w$

is eliminated. As a result, the optimal family tax deduction rate and the basic income are close, regardless of the tax rate.

The simplest formulation of the family tax deduction is as follows. Let $\psi > 0$ be the child tax deduction rate, i.e. having $n$ children, amount $\psi n$ can be deducted from the proportional personal income tax $\theta w$, up to the maximum $\theta w$. To avoid absurd cases, we assume that the family tax deduction is always lower than or equal to the narrow raising cost: $\psi = \pi z$. Let $t_0$ denote now the tax deducted: $t_0 = \min(\theta w, \psi n)$. Obviously, if the benefit is so low or the tax rate is so high that even the minimal wage earner's tax amount is higher than the family tax deduction, then the latter reduces to the child benefit. But this has already been covered in Section 3, therefore we assume that $\theta w_m \le \psi n$, where $w_m$ is the minimal wage.

By definition, type $w$'s own consumption is equal to

$$c = (z + t_0)(1 - \pi n). \tag{13}$$

We have now two domains; *slack*, denoted by $S$: $\theta w_S > \psi n_S$ and *tight*, denoted by $T$: $\theta w_T 0 \le \psi n_T$. (The status of the demarcation line $\theta w = \psi n$ is ambiguous.) Correspondingly, $t_{0S} = \psi n$ and $t_{0T} = \theta w$. Then (13) branches off into

$$c_S = (z_S + \psi n_S)(1 - \pi n_S), \qquad z_S = \hat{\theta} w_S + \gamma \tag{13$S$}$$

and

$$c_T = (w_T + \gamma)(1 - \pi n_T). \tag{13$T$}$$

Then there are two separate regimes with their own fertility optima. Lemma 2 provides $n_S$ for $\psi$ replacing $\varphi$ in (7)–(8), while $n_T$ for $\varphi = 0$ and $\theta = 0$, i.e. ($7^{\text{o}}$). It can be shown that $n_S(w) > n_T$. It is especially disturbing that the transition from $S$ into $T$ is discontinuous: the optimal transfer drops a lot due to a minor tax rate rise!

To formulate the new balance condition, we repeat the argument leading to (9). But now we take into account the partition along $S - T$, which depends on the parameter vector $(w, \psi, \gamma)$. For convenience, we assume that the relevant functions $n_S(w)$ and $n_T(w)$ are also defined outside their natural domains, being equal to zero outside their proper domains. Subindexes $S$ and $T$ refer to these restricted expectations. The reformulated balance equation is as follows:

$$\theta = \psi \mathbf{E}_S n_S(w, \psi, \gamma) + \theta \mathbf{E}_T w + \gamma. \tag{14}$$

Our social welfare function remains basically the same as above, only $\psi$ replaces $\varphi$.

Due to its simplicity, it is worth discussing the two-type case of family tax deduction.

**Example 4.** In the two-type case, the low-wage type is tight, the high-wage type is slack. Furthermore, at the social optimum, the low wage type's family tax deduction is equal to her tax: $\psi^* n_L = \theta w_L$. Inserting ($7^{\text{o}}$), our optimality condition becomes

$$\psi^* = \frac{\bar{\zeta} \pi \theta w_L}{\zeta}. \tag{15}$$

The balance condition (14) can be simplified by making the following observation: since the low-wage type deducts all her tax obligation, therefore the high-wage type's tax is used for financing her own family tax deduction and both types' basic income:

$$f_H \theta w_H = f_H \psi n(w_H, \psi, \gamma) + \gamma. \tag{14$'$}$$

Substituting (15) into (14') and using (7)–(8) yield an equation for $\gamma$, regardless of the paternalistic preference parameter $\zeta^*$.

It is easy to grasp that in general the optimal family tax deduction is far from being socially optimal. Due to the elimination of the excess transfer, the low-wage type's fertility is as low as $n_0$ in (7°) and the corresponding net income is only $y_L = w_L + \gamma$, which is lower than in the pure basic income system: $y_L^{\mathrm{o}} = \hat{\theta} w_L + \theta$. The high-wage type's fertility is higher than $n_0$ but the additional resource brings less gain in her consumption and child welfare than the loss is in the low-wage type's welfare.

We continue the numerical illustrations. Having a corner solution (at least in our two-type model), now the socially optimal tax deduction rate and basic income are independent of the key parameter of the social welfare function, $\zeta^*$. We choose again three parameter values: $\zeta^* = 0.4$, 0.5 and 0.6 which still influence the numerical value of the efficiency. According to Table 2, the two socially optimal transfer rates are close to each other for any tax rate. For example, at $\theta = 0.3$, the tax deduction rate $\psi^*$ and the basic income $\gamma^*$ are equal to 0.18 and 0.14, respectively. The rise of fertility is much weaker than with the child benefits, but higher than in the pure basic income system. The child benefit system is more efficient than the family tax deduction system, which in turn is about as efficient as the pure basic income system.

**Table 2.** *Impact of the optimal family tax deduction rate for various $\theta$s and $\zeta^*$s*

| Tax rate $\theta$ | Family tax deduction r. $\psi^*$ | Basic income $\gamma^*$ | Fertility: Low wage $n_L$ | Fertility: High wage $n_H$ | Fertility: Average $n$ | Relative efficiency: No paternalism $\varepsilon_N$ | Relative efficiency: Weak paternalism $\varepsilon_W$ | Relative efficiency: Strong paternalism $\varepsilon_S$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0.816 | 0.816 | 0.816 | 1 | 1 | 1 |
| 0.1 | 0.061 | 0.049 | 0.816 | 0.856 | 0.829 | 0.999 | 1.001 | 1.002 |
| 0.2 | 0.123 | 0.097 | 0.816 | 0.902 | 0.845 | 0.997 | 1.000 | 1.003 |
| 0.3 | 0.184 | 0.142 | 0.816 | 0.955 | 0.863 | 0.992 | 0.997 | 1.002 |

## 4. Conclusions

In our very simple model, assuming a pronatalistic government, the socially optimal child benefit system achieves higher efficiency than the family tax deduction system and even the pure basic income system may overtake the deduction system for a progressive social welfare function, not discussed here.

We warn the reader on the limits of the model. We have used the simplest utility function pair, two logarithmic functions. Even at the modest generalization into CRRA, the independence of the fertility of the wage in Example 1 (no child benefit) disappears, therefore the saving of Theorems 1 and 2 requires further nontrivial assumptions. The neglect of the negative impact of taxation on labor supply and tax reporting further weakens the force of our numerical examples. The inclusion of labor disutility and flexible labor supply would make the model more realistic and determine the optimal tax rate, would more fully highlight the differences between the two transfer systems.

But this modification would complicate the analysis, therefore we have no used it here. The heterogeneity of the relative raising cost also deserves an examination.

## Appendix: Errors of linearization

In this Appendix, we shall present errors stemming from two linearizations—potentially or actually used in the literature. We mostly concentrate on the quantitative errors but the qualitative ones are also important.

### 1) Single linearization

There is always a temptation to simplify things, in our case, to linearize the adult consumption equation (2):

$$c_1 = z - (\pi z - \varphi_1)n, \quad \text{where} \quad z = \hat{\theta} w + \gamma. \tag{A.1-1}$$

Here $c_1 > c$, therefore the approximate optimal fertility is higher than the true one. Then (5) is replaced by

$$0 = u'(n) = -\frac{\pi z - \varphi_1}{z + (\pi z - \varphi_1)n} + \frac{\zeta}{n}. \tag{A.1-2}$$

The optimal fertility is now simply

$$n_1(w, \varphi_1, \gamma) = \frac{\zeta z}{\bar{\zeta}(\pi z - \varphi_1)}. \tag{A.1-3}$$

Note that for our upper bound $\varphi_1 = \pi z_m$ mentioned in Example 1, the fertility is infinite: the approximation breaks down.

Until now we have neglected the balance condition. To simplify the calculations, we only consider the case of uniform wages: $w \equiv 1$ and take the basic income zero: $\gamma = 0$. Then $z = \hat{\theta}$ and the balance equation reduces to $\theta = \varphi_1 n_1(\theta, \varphi_1)$. Inserting (A.1–3) into the balance equation results in

$$\theta = \varphi_1 \frac{\zeta \hat{\theta}}{\bar{\zeta}(\hat{\theta}\pi - \varphi_1)}, \quad \text{i.e.} \quad \varphi_1(\theta) = \frac{\bar{\zeta}\theta\hat{\theta}\pi}{\zeta\hat{\theta} + \bar{\zeta}\theta}. \tag{A.1-4}$$

### 2) Double linearization

Earlier papers (GLM and Simonovits, 2013) not only linearized the consumption equation but also made the raising costs independent of the net income:

$$c_2 = z - (\pi - \varphi_2)n. \tag{A.2-1}$$

Incidentally, that way the error became smaller, because the elimination of the term $-\pi z \varphi_1 n^2$ from (2) was countered by the deduction of $\pi(1 - z)n$, which is positive for GLM's $\gamma = 0$ and $w = 1$, namely $\pi\theta n$.

Returning to the general case, the optimal fertility (A.1–3) changes into

$$n_2(w, \varphi_2, \gamma) = \frac{\zeta z}{\bar{\zeta}(\pi - \varphi_2)}. \qquad (A.2-3)$$

Comparing (A.2–3) with (A.1–3), we see that for low-wage types, $z < 1$, therefore for $\varphi_1 = \varphi_2$, the fertility in (A.2–3) is lower than in (A.1–3) and vice versa.

Returning to uniform wages, the balance condition now reduces to

$$\theta = \varphi_2 \frac{\zeta \hat{\theta}}{\bar{\zeta}(\pi - \varphi_2)}, \quad \text{i.e.} \quad \varphi_2(\theta) = \frac{\bar{\zeta}\theta\pi}{\zeta\hat{\theta} + \bar{\zeta}\theta} \qquad (A.2-4)$$

etc. Comparing (A.1–4) and (A.2–4) implies $\varphi_1(\theta) = \hat{\theta}\varphi_2(\theta) < \varphi_2(\theta)$.

### Numerical comparison

Rather than struggling with analytical comparisons, we present our numerical results in Table A.1. In column 2 the exact schedule $\varphi(\theta)$ is shown: while the tax rate rises from 0.1 to 0.3, the benefit rate rises from 0.105 to 0.255. Column 3 displays the corresponding fertility. In columns 4 and 5 (single linearization) the benefit rate and the fertility are shown, respectively. As the tax rate rises, the benefit rate lags more and more behind the correct value: $0.147 < 0.255$, while the fertility overtakes the correct one. In columns 6 and 7 (double linearization) the errors partly cancel each other: for $\theta = 0.3$, the new benefit rate 0.21 lies just around halfway between the correct (0.255) and the approximate (0.147) values.

**Table A.1.** *Nonlinear vs. linear raising cost equations*

| Tax | Nonlinear | | Linear 1 | | Linear 2 | |
|---|---|---|---|---|---|---|
| rate | benefit r. | fertility | benefit r. | fertility | benefit r. | fertility |
| $\theta$ | $\varphi$ | $n$ | $\varphi_1$ | $n_1$ | $\varphi_2$ | $n_2$ |
| 0.1 | 0.105 | 0.952 | 0.088 | 1.134 | 0.098 | 1.020 |
| 0.2 | 0.186 | 1.071 | 0.131 | 1.531 | 0.163 | 1.224 |
| 0.3 | 0.255 | 1.176 | 0.147 | 2.041 | 0.210 | 1.429 |

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