MAGYAR TUDOMÁNYOS AKADÉMIA
Közgazdaság- és Regionális Tudományi Kutatóközpont

MTA KRTK KTI

Centre for Economic and Regional Studies
HUNGARIAN ACADEMY OF SCIENCES

MŰHELYTANULMÁNYOK DISCUSSION PAPERS

MT-DP – 2015/12

Universal Characterization Sets for the Nucleolus in Balanced Games

TAMÁS SOLYMOSI - BALÁZS SZIKLAI

INSTITUTE OF ECONOMICS, CENTRE FOR ECONOMIC AND REGIONAL STUDIES,
HUNGARIAN ACADEMY OF SCIENCES BUDAPEST, 2015

Discussion papers
MT-DP – 2015/12

Institute of Economics, Centre for Economic and Regional Studies,
Hungarian Academy of Sciences

Universal Characterization Sets for the Nucleolus in Balanced Games

Authors:

Tamás Solymosi
senior research fellow
Momentum Game Theory Research Group
Institute of Economics - Centre for Economic and Regional Studies
Hungarian Academy of Sciences
e-mail: solymosi.tamas@krtk.mta.hu

Balázs Sziklai
junior research fellow
Momentum Game Theory Research Group
Institute of Economics - Centre for Economic and Regional Studies
Hungarian Academy of Sciences
e-mail: sziklai.balazs@krtk.mta.hu

February 2015

ISBN 978-615-5447-71-6
ISSN 1785 377X

Universal Characterization Sets for the Nucleolus in Balanced Games

Tamás Solymosi - Balázs Sziklai

Abstract

We provide a new modus operandi for the computation of the nucleolus in cooperative games with transferable utility. Using the concept of dual game we extend the theory of characterization sets. Dually essential and dually saturated coalitions determine both the core and the nucleolus in monotonic games whenever the core is non-empty. We show how these two sets are related with the existing characterization sets. In particular we prove that if the grand coalition is vital then the intersection of essential and dually essential coalitions forms a characterization set itself. We conclude with a sample computation of the nucleolus of bankruptcy games - the shortest of its kind.

Keywords: Cooperative game theory, Nucleolus, Characterization sets

JEL classification: C71

Univerzális karakterizációs halmazok a nukleolusz kiszámítására kiegyensúlyozott játékokon

Solymosi Tamás – Sziklai Balázs

Összefoglaló

Műhelytanulmányunkban egy új általános megközelítést adunk arra nézve, hogyan lehet a nukleoluszt kiszámolni kooperatív átváltható hasznosságú játékok esetén. A duál játék fogalmát felhasználva két új karakterizációs halmazt definiálunk. A duálisan lényeges és a duálisan telített koalíciók mind a magot, mind a nukleoluszt meghatározzák monoton kiegyensúlyozott játékok esetén. Elemezzük, hogy az újonnan bevezetett karakterizációs halmazok milyen kapcsolatban állnak a már meglévőkkel. Speciálisan megmutatjuk, hogy ha a nagykoalíció erősen lényeges (vitális), akkor a lényeges és a duálisan lényeges koalícióknak a metszete maga is egy karakterizációs halmaz. Végezetül a csődjáték nukelouszának a kiszámításával demonstráljuk az elméleti eredmények alkalmazhatóságát.

Tárgyszavak: kooperatív játékok, nukleolusz, karakterizációs halmazok

JEL kód: C71

Universal Characterization Sets for the Nucleolus in Balanced Games

Tamás Solymosi   Balázs Sziklai

February 6, 2015

Abstract

We provide a new modus operandi for the computation of the nucleolus in cooperative games with transferable utility. Using the concept of dual game we extend the theory of characterization sets. Dually essential and dually saturated coalitions determine both the core and the nucleolus in monotonic games whenever the core is non-empty. We show how these two sets are related with the existing characterization sets. In particular we prove that if the grand coalition is vital then the intersection of essential and dually essential coalitions forms a characterization set itself. We conclude with a sample computation of the nucleolus of bankruptcy games - the shortest of its kind.

Keywords: Cooperative game theory, Nucleolus, Characterization sets

JEL-codes: C71

1 Introduction

The nucleolus, developed by Schmeidler (1969), soon became one of the most frequently applied solution concepts of cooperative game theory. Despite its good properties it lost some popularity in the last 20 or so years. Very much like the Shapley-value it suffers from computational difficulties. While the former has an explicit formula and various axiomatizations, the nucleolus can only be computed by an LP and its axiomatization is less straightforward.

Computing the nucleolus is a notoriously hard problem, even $\mathcal{NP}$-hard for some classes of games. While $\mathcal{NP}$-hardness was proven for minimum cost spanning tree games (Faigle, Kern, and Kuipers, 1998), voting games (Elkind, Goldberg, Goldberg, and Wooldridge, 2009) and flow and linear production games (Deng, Fang, and Sun, 2009), it is still unknown whether the corresponding decision problem – i.e. verifying whether an allocation is the nucleolus or not – belongs to $\mathcal{NP}$ or not.

In recent years several polynomial time algorithms were proposed to find the nucleolus of important families of cooperative games, like standard tree, assignment, matching and bankruptcy games (Maschler, Potters, and Reijnierse, 2010; Solymosi and Raghavan, 1994; Kern and Paulusma, 2003; Aumann and Maschler, 1985). In addition Kuipers (1996) and Arin and Inarra (1998) developed methods to compute the nucleolus for convex games.

The main breakthrough came from another direction. In their seminal paper Maschler, Peleg, and Shapley (1979) described the geometric properties of the nucleolus and devised a computational framework in the form of a sequence of linear programs. Although these LPs consist of exponentially many inequalities they can be solved efficiently if one knows which constraints are redundant. Huberman (1980); Granot, Granot, and Zhu (1998); Reijnierse and Potters (1998) provided methods to identify coalitions that correspond to non-redundant constraints.

Granot, Granot, and Zhu (1998) provided the most fruitful approach. They introduced the concept of characterization set which is a collection of coalitions that determines the nucleolus by itself. They proved that if the size of the characterization set is polynomially bounded in the number of players, then the nucleolus of the game can be computed in strongly polynomial time. A collection that characterizes the nucleolus in one game need not characterize it in another one. Thus we are interested in characterization sets that are universal, i.e. that yield the nucleolus in every TU-game.

Huberman (1980) was the first to show that such a collection exists. He introduced the concept of essential coalitions which are coalitions that have no weakly minorizing partition. Granot, Granot, and Zhu (1998) provided another collection that characterize the nucleolus in cost games with non-empty cores. Saturated coalitions contain all the players that can join the coalition without imposing extra cost.

We introduce two new characterization sets: dually essential and dually saturated coalitions. We show that each dually inessential coalition has a weakly minorizing overlapping decomposition which consists exclusively of dually essential coalitions. Thus dually essential coalitions determine the core, and if the core is non-empty they determine the nucleolus as well. If every player contributes to the value of a coalition then such coalition is called dually saturated. We show that dually saturated coalitions also determine the core, and if the core is non-empty, then also the nucleolus of a TU-game.

The larger a characterization set is the easier to uncover it in a particular game class. However with smaller characterization set it comes a faster LP. Hence there is a tradeoff between the difficulty in identifying the members of a characterization set and its efficiency. In order to exploit this technique we analyze the relationship of the four known universal characterization sets. We prove that essential coalitions are a subset of dually saturated coalitions in monotonic profit games and that dually essential coalitions are a subset of saturated coalition in case of monotonic cost games. We show that in general essential and dually essential coalitions do not contain each other. In fact for additive games their intersection is trivial (consist of the grand coalition only). We prove that if the grand coalition is vital then the intersection of essential and dually essential coalitions forms a characterization set itself.

2 Game theoretical framework

A *cooperative game with transferable utility* is an ordered pair (N, v) consisting of the player set $N = \{1, 2, \ldots, n\}$ and a characteristic function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. The value $v(S)$ represents the worth of coalition S. No matter how other players behave if the players of S work together they can secure themselves $v(S)$ amount of payoff. The set N – when viewed as a coalition – is called the grand coalition.

Definition 1. *A cooperative game (N, v) is called* monotonic *if*

$$S \subseteq T \subseteq N \;\Rightarrow\; v(S) \leq v(T).$$

and superadditive *if*

$$(S, T \subset N,\; S \cap T = \emptyset) \;\Rightarrow\; v(S) + v(T) \leq v(S \cup T).$$

In superadditive games two disjoint coalitions can always merge without losing money. Hence we shall assume that players form the grand coalition. The main question is then how to distribute $v(N)$ among the players in some fair way.

A solution for a cooperative game $\Gamma = (N, v)$ is a vector $x \in \mathbb{R}^N$ that represents the payoff of each player. For convenience, we introduce the following notations $x(S) = \sum_{i \in S} x_i$ for any $S \subseteq N$, and instead of $x(\{i\})$ we simply write $x(i)$. A solution is called *efficient* if $x(N) = v(N)$ and *individually rational* if $x(i) \geq v(i)$ for all $i \in N$. The imputation set of the game $I(\Gamma)$ consists of the efficient and individually rational solutions, formally,

$$I(\Gamma) = \{x \in \mathbb{R}^N \mid x(N) = v(N),\; x(i) \geq v(i) \text{ for all } i \in N\}.$$

Given an allocation $x \in \mathbb{R}^N$, we define the *satisfaction* of a coalition S as

$$sat_\Gamma(S, x) := x(S) - v(S).$$

The core of the cooperative game $\mathcal{C}(\Gamma)$ is a set-valued solution where all the satisfaction values are non-negative. Formally,

$$\mathcal{C}(\Gamma) = \{x \in \mathbb{R}^N \mid x(N) = v(N),\ x(S) \geq v(S) \text{ for all } S \subseteq N\}.$$

Definition 2. *A cooperative game* (N, v) *is* convex *if the characteristic function is supermodular i.e.*

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T), \quad \forall\ S, T \subseteq N.$$

Deriving results for convex games is a less challenging task than in general due to their nice structure. Here we only mention a result of **?**, namely that the core of a convex game is not empty.

In many economic situations cooperation results in cost saving rather than profit growth. For instance such situation occurs when customers would like to gain access to some public service or public facility. The question is then, how to share the costs of the service. Cost allocation games can be modeled in a similar fashion as cooperative TU-games.

A *cooperative cost game* is an ordered pair (N, c) consisting of the player set $N = \{1, 2, \ldots, n\}$ and a characteristic cost function $c : 2^N \to \mathbb{R}$ with $c(\emptyset) = 0$. The value $c(S)$ represent how much cost coalition S must bear if it chooses to act separately from the rest of the players. In most cases c is monotonic and subadditive. That is the more people use the service the more it costs, however there is also an increase of efficiency.

Definition 3. *A cost game* (N, c) *is called* subadditive *if*

$$(S, T \subset N,\ S \cap T = \emptyset)\ \Rightarrow\ c(S) + c(T) \geq c(S \cup T).$$

and concave *if,*

$$c(S) + c(T) \geq c(S \cup T) + c(S \cap T), \quad \forall\ S, T \subseteq N.$$

It is possible to associate a profit game (N, v) with a cost game (N, c), called the *savings game*, which is given by $v(S) = \sum_{i \in S} c(i) - c(S)$ for all $S \subseteq N$. Note that a cost game is subadditive (concave) if and only if the corresponding savings game is superadditive (convex). Similarly the relationship is reversed in all the previously listed concepts such as individual rationality, satisfaction, the core and so on. Formally, let $\Gamma = (N, c)$ be a cost game and $x \in \mathbb{R}^N$ an arbitrary allocation. The *satisfaction* of a coalition S is defined as

$$sat_\Gamma(S, x) := c(S) - x(S).$$

As we pointed out in case of cost games the order of the characteristic function and the payoff vector is reversed, hence the formula for sat_Γ depends on whether Γ is a cost or profit game. However, the essence of the $sat_\Gamma(S,x)$ expression does not change. It still indicates the contentment of coalition S under payoff vector x.

In a cost game (N,c) we say that x is individually rational if $x(i) \leq c(i)$ for all $i \in N$ and x is a core member if all the satisfaction values are non-negative. Notice that core vectors of monotonic profit and cost games are non-negative. Indeed,

$$x(i) = x(N) - x(N \setminus i) \geq c(N) - c(N \setminus i) \geq 0$$

for any core allocation x and $i \in N$. For profit games this is even more trivial since $x(i) \geq v(i) \geq v(\emptyset) = 0$.

We say that a vector $x \in \mathbb{R}^m$ *lexicographically precedes* $y \in \mathbb{R}^m$ (denoted by $x \preceq y$) if either $x = y$ or there exists a number $1 \leq j < m$ such that $x_i = y_i$ if $i < j$ and $x_j < y_j$. Let $\Gamma = (N, v)$ be a game and let $\theta(x) \in \mathbb{R}^{2^n}$ be the satisfaction vector that contains the 2^n satisfaction values in a non-decreasing order.

Definition 4. *The* nucleolus of Γ with respect to X *is the set of allocations of a game $x \in \mathbb{R}^n_+$ that lexicographically maximizes $\theta(x)$ over X. Formally,*

$$\mathcal{N}(\Gamma, X) = \{x \in X \mid \theta(y) \preceq \theta(x) \textit{ for all } y \in X\}.$$

It is well known that if X is nonempty and compact then $\mathcal{N}(\Gamma, X) \neq \emptyset$ and if X is convex then $\mathcal{N}(\Gamma, X)$ consist of a single point (for proof see (Schmeidler, 1969)). Furthermore the nucleolus is a continuous function of the characteristic function. If X is chosen to be the set of allocations, we speak of the *prenucleolus of* Γ, if X is the set of imputations then we speak of the *nucleolus of* Γ. Throughout the paper we will use the shorthand notation $\mathcal{N}(\Gamma)$ for $\mathcal{N}(\Gamma, I(\Gamma))$.

3 Characterization sets

The concept of characterization sets was already used by Megiddo (1974), but somehow went unnoticed at that time. Later Granot, Granot, and Zhu (1998) and Reijnierse and Potters (1998) re-introduced the idea almost simultaneously. It is remarkable that two such closely related and revolutionary papers appeared in the same year. Here we will use the formalism of Granot, Granot, and Zhu (1998).

Definition 5. *Let $\Gamma^{\mathcal{F}} = (N, \mathcal{F}, v)$ be a cooperative game with coalition formation restrictions, where $\mathcal{F} \subseteq 2^N$ consists of all coalitions deemed permissible. Then $\mathcal{F}$ is called a characterization set for the nucleolus of the game $\Gamma = (N, v)$, if $\mathcal{N}(\Gamma^{\mathcal{F}}) = \mathcal{N}(\Gamma)$.*

The main result of (Granot, Granot, and Zhu, 1998) is presented the following theorem. We denote by $e_S \in \{0,1\}^N$ the membership vector of coalition S given by $(e_S)_i = 1$ if $i \in S$ and $(e_S)_i = 0$ otherwise.

Theorem 6. *Let $\Gamma = (N, g)$ be a cooperative game, where g is either a characteristic cost or profit function and let $\mathcal{F} \subset 2^N$. Denote by x the nucleolus of $\Gamma^{\mathcal{F}}$. The collection $\mathcal{F}$ is a characterisation set for the nucleolus of Γ if for every $S \in 2^N \setminus \mathcal{F}$ there exists a nonempty subcollection $\mathcal{F}_S$ of $\mathcal{F}$, such that*

i. $sat_\Gamma(T, x) \leq sat_\Gamma(S, x)$, *whenever $T \in \mathcal{F}_S$,*

ii. *e_S can be expressed as a linear combination of $\{e_T : T \in \mathcal{F}_S\}$.*

Unfortunately the direction can not be reversed, i.e., the above conditions are sufficient but not at all necessary. Take for example the (superadditive, but not balanced) profit game with four players $N = \{1, 2, 3, 4\}$ and the following characteristic function: $v(i) = 0$, $v(i, j) = 1$, $v(i, j, k) = 4$ for any $i, j, k \in N$ and let $v(N) = 4$. Then the 2-player coalitions and the grand coalition are sufficient to determine the nucleolus, which is given by $z(i) = 1$ for all $i \in N$. However the 3-player coalitions have smaller satisfaction values at z, thus the first condition of Theorem 6 is violated. Notice that in this game the 3-player coalitions and the grand coalition are also sufficient to determine the nucleolus.

In general neither the 2-player nor the 3-player coalitions (and the grand coalition) characterize the nucleolus. The fact that in this example they did was due to the particular choice (the symmetry) of the coalitional function. We would like to deal with collections that characterize the nucleolus independently of the realization of the coalitional function. We say that a characterization set $\mathcal{F}$ is *universal* for a class of games if it satisfies both conditions of Theorem 6 in every game from that class of games. Special focus will be given to the class of games with a non-empty core.

A straightforward corollary of Theorem 6 is that we can enlarge universal characterisation sets arbitrarily.

Corollary 7. *Let $\mathcal{F} \subset 2^N$ be a characterisation set that satisfies both conditions of Theorem 6. Then $\mathcal{T}$ is a characterisation set for any $\mathcal{F} \subset \mathcal{T} \subseteq 2^N$*

Now we present four universal characterization sets for balanced games. The first one is due to Huberman (1980).

Definition 8 (Essential coalitions)**.** *Let N be a set of players, (N, v) a profit, (N, c) a cost game. Coalition S is called* essential *in game $\Gamma = (N, v)$ if it can not be partitioned as $S = S_1 \dot\cup \ldots \dot\cup S_k$ with $k \geq 2$ such that*

$$v(S) \leq v(S_1) + \ldots + v(S_k).$$

Similarly S is called essential in game $\Gamma = (N, c)$ if it can not be partitioned as $S = S_1 \dot\cup \ldots \dot\cup S_k$ with $k \geq 2$ such that

$$c(S) \geq c(S_1) + \ldots + c(S_k).$$

The set of essential coalitions is denoted by $\mathcal{E}(\Gamma)$, where Γ is either (N, v) or (N, c).

By definition, the singleton coalitions are always essential in every game. It is easily seen that in a profit / cost game each non-essential coalition has a weakly majorizing / minorizing partition which consists exclusively of essential coalitions. Such coalitions are called *inessential*. Moreover, the core is determined by the efficiency equation $x(N) = v(N)$ and in profit games the $x(S) \geq v(S)$ inequalities, while in cost games the $x(S) \leq c(S)$ inequalities corresponding to the essential coalitions, all the other inequalities can be discarded from the core system.

Huberman (1980) showed that if the core of the game is non-empty then the grand coalition and the essential coalitions form a characterization set for the nucleolus. This observation helps us to eliminate large coalitions which are redundant for the nucleolus. To detect small coalitions that are unnecessary for the nucleolus, we need the concept of dual game.

Definition 9. *The* dual game (N, v^*) *of game* (N, v) *is defined by the coalitional function* $v^*(S) := v(N) - v(N \setminus S)$ *for all* $S \subseteq N$.

Clearly, $v^*(\emptyset) = 0$, hence (N, v^*) is indeed a cooperative TU-game. Notice that $v^*(N) = v(N)$ and $(v^*)^*(S) = v(S)$ for all $S \subseteq N$. It will be useful to think of the dual game of a profit game as a cost game and vice versa. We can identify the small redundant coalitions, if we apply Huberman's argument to the dual game.

Definition 10 (Dually essential coalitions). *Let N be a set of players, (N, v) a profit, (N, c) a cost game. Coalition S is called* dually essential *in game (N, v) if its complement can not be partitioned as $N \setminus S = (N \setminus T_1) \dot{\cup} \ldots \dot{\cup} (N \setminus T_k)$ with $k \geq 2$ such that*

$$v^*(N \setminus S) \geq v^*(N \setminus T_1) + \ldots + v^*(N \setminus T_k),$$

or equivalently,

$$v(S) \leq v(T_1) + \ldots + v(T_k) - (k-1)v(N).$$

Similarly, S is called dually essential *in cost game (N, c) if its complement can not be partitioned as $N \setminus S = (N \setminus T_1) \dot{\cup} \ldots \dot{\cup} (N \setminus T_k)$ with $k \geq 2$ such that*

$$c^*(N \setminus S) \leq c^*(N \setminus T_1) + \ldots + c^*(N \setminus T_k),$$

or equivalently,

$$c(S) \geq c(T_1) + \ldots + c(T_k) - (k-1)c(N).$$

The set of dually essential coalitions is denoted by $\mathcal{DE}(\Gamma)$, where Γ is either (N, v) or (N, c).

Notice that each member of S appears in all of the coalitions T_1, ..., T_k, but every other player appears exactly $k-1$ times in this family. We call such a system of coalitions an *overlapping decomposition* of S. For a more general definition, where the complements of the overlapping coalitions need not form a partition of the complement coalition, see e.g. Brânzei, Solymosi, and Tijs (2005) and the references therein.

By definition, all $(n-1)$-player coalitions are dually essential in any game. It is easily checked that if S and T are not dually essential coalitions and T appears in an overlapping decomposition of S, then S cannot appear in an overlapping decomposition of T. Consequently, in a profit / cost game each dually non-essential (*dually inessential*) coalition has a weakly majorizing / weakly minorizing overlapping decomposition which consists exclusively of dually essential coalitions. Moreover, the core of profit game (N, v) can also be determined by the dual efficiency equation $x(N) = v^*(N)$ and the $x(S) \leq v^*(S)$ dual inequalities corresponding to the complements of the dually essential coalitions, all the other dual inequalities can be discarded from the dual core system. An analogous statement holds for the core of cost games.

The main feature of dually essential coalitions for the nucleolus lies in the next theorem.

Theorem 11. *If $\mathcal{C}(\Gamma) \neq \emptyset$, then the grand coalition and the dually essential coalitions form a characterization set for $\mathcal{N}(\Gamma)$.*

Proof. There are many ways to derive this result. A formal proof can be obtained by copying the arguments in Huberman (1980). Here we pursue another way and deduce it from Theorem 6.

Let S be a dually inessential coalition in the balanced profit game $\Gamma = (N, v)$. As remarked earlier, S has a weakly minorizing overlapping decomposition $T_1, \ldots, T_k$ $(k \geq 2)$ which consists exclusively of dually essential coalitions. Hence **ii**. of Theorem 6 follows immediately. The first part follows from the fact that in balanced games the nucleolus is in the core, and for any $x \in \mathcal{C}(\Gamma)$

$$\begin{aligned} v(S) &\leq v(T_1) + \ldots + v(T_k) - (k-1)v(N) \\ v(S) - x(S) &\leq v(T_1) + \ldots + v(T_k) - (k-1)x(N) - x(S) \\ -sat_\Gamma(S, x) &\leq -(sat_\Gamma(T_1, x) + \ldots + sat_\Gamma(T_k, x)) \\ sat_\Gamma(S, x) &\geq sat_\Gamma(T_1, x) + \ldots + sat_\Gamma(T_k, x) \geq 0, \end{aligned}$$

where the second inequality comes from $v(N) = x(N)$, while the third from the identity $x(T_1) + \ldots + x(T_k) = (k-1)x(N) + x(S)$ implied by $N \setminus S = (N \setminus T_1) \dot{\cup} \ldots \dot{\cup} (N \setminus T_k)$. The satisfaction values are non-negative for any core allocation x, hence $sat_\Gamma(S, x) \geq sat_\Gamma(T_j, x)$ for all $j = 1, \ldots, k$. □

The next characterization set was proposed by Granot, Granot, and Zhu (1998) for monotonic balanced cost games.

Definition 12. *A coalition S is said to be* saturated *in cost game (N, c) if $i \in S$ whenever $c(S) = c(S \cup \{i\})$.*

In other words if S is a saturated coalition then every new member will impose extra cost on the coalition. A saturated coalition S said to be *irreducible* if there is no partition $S_1, \ldots, S_k$ of S such that S_i are saturated and $c(S) \geq c(S_1) + \ldots + c(S_k)$. Let $\mathcal{S}^*(\Gamma)$ denote the set of all irreducible saturated coalitions and

$$\mathcal{S}(\Gamma) = \mathcal{S}^*(\Gamma) \cup \{N \setminus i \mid i \in N\} \cup \{N\}.$$

Theorem 13. *(Granot, Granot, and Zhu, 1998) Let $\Gamma = (N, c)$ be a monotonic cost game with a non-empty core, then $\mathcal{S}(\Gamma)$ forms a characterization set for $\mathcal{N}(\Gamma)$.*

Similarly to the other characterization sets, $\mathcal{S}(\Gamma)$ also induces a representation of the core $\mathcal{C}(\Gamma)$ as well. Let us mention here that just because a collection of coalitions determines the core it does not necessarily characterize the nucleolus of the game. Maschler, Peleg, and Shapley (1979) presented two games with the same core, but with different nucleoli.

We now convert the concept of saturatedness to monotonic profit games based on the dualization correspondence between profit and cost games. Let (N, v) be a monotonic profit game and $S \subseteq N$ be an arbitrary coalition. We say that S is *dually saturated* if $v(S \setminus i) < v(S)$ for any $i \in S$. In other words every member contributes to the worth of coalition S. A dually saturated coalition S said to be *irreducible* if there is no partition $S_1, \ldots, S_k$ of S such that S_i are dually saturated and $v(S) \leq v(S_1) + \ldots + v(S_k)$. Let $\mathcal{DS}^*(\Gamma)$ denote the set of all irreducible dually saturated coalitions and

$$\mathcal{DS}(\Gamma) = \mathcal{DS}^*(\Gamma) \cup \{i \mid i \in N\} \cup \{N\}.$$

The following definition is needed for our next theorem. Let (N, v) be a monotonic game and $S \subseteq N$ a dually non-saturated coalition, then we say that $\underline{S} \neq \emptyset$ is a lower closure of S if $\underline{S} \subset S$, $v(\underline{S}) = v(S)$ and $\underline{S}$ is a dually saturated coalition. Note that if S has no lower closure, then no member contributes to the worth of S or to any subset of S. Hence $v(S) = v(i) = v(\emptyset) = 0$ for any $i \in S$.

Theorem 14. *Let $\Gamma = (N, v)$ be a monotonic game with a non-empty core, then $\mathcal{DS}(\Gamma)$ forms a characterization set for $\mathcal{N}(\Gamma)$.*

Proof. Again we will use Theorem 6. Let S be a dually saturated but not irreducible coalition, then there exists a partition $S_1, \ldots, S_k$ of S, such that $v(S) \leq v(S_1) + \ldots + v(S_k)$. Hence $sat_\Gamma(S, x) \geq sat_\Gamma(S_1, x) + \ldots + sat_\Gamma(S_k, x)$. Note that we can choose $S_1, \ldots, S_k$ to be irreducible, since if one of them is not, then we take an irreducible refinement of it. Thus $S_1, \ldots, S_k \in \mathcal{DS}(\Gamma)$.

Now let S be a dually non-saturated coalition. If S has no lower closure then $v(S) = v(i) = 0$ for any $i \in S$. From this observation also follows that $sat_\Gamma(\{i\}, x) \leq sat_\Gamma(S, x)$ for any $i \in S$ and for any allocation x. Since all the singleton coalitions are included in $\mathcal{DS}(\Gamma)$ by Theorem 6, S can be discarded. Finally let $\underline{S}$ be a lower closure of S and let $S \setminus \underline{S} = T$, then

$$sat_\Gamma(\underline{S}, x) + x(T) = x(\underline{S}) + x(T) - v(\underline{S}) = x(S) - v(S) = sat_\Gamma(S, x).$$

Since core vectors are non-negative this also means $sat_\Gamma(\underline{S}, x) \leq sat_\Gamma(S, x)$ for any $x \in \mathcal{C}(\Gamma)$. Now we show that $sat_\Gamma(\{i\}, x) \leq sat_\Gamma(S, x)$ for any $i \in T$.

$$\begin{aligned} sat_\Gamma(\{i\}, x) = x(i) - v(i) \leq x(i) = \\ x(S) - x(S \setminus i) + v(S \setminus i) - v(S) = \\ sat_\Gamma(S, x) - sat_\Gamma(S \setminus i, x) \leq sat_\Gamma(S, x) \end{aligned}$$

We have shown that for any $S \in 2^N \setminus \mathcal{DS}(\Gamma)$ there exist a subcollection $\mathcal{F}$ of $\mathcal{DS}(\Gamma)$, such that $\mathcal{F}$ fulfills both conditions of Theorem 6. Hence $\mathcal{DS}(\Gamma)$ is a characterization set for $\mathcal{N}(\Gamma)$. □

Next we show a relationship between dually essential and saturated coalitions.

Lemma 15. *Let $\Gamma = (N, c)$ be a monotonic cost game, then $\mathcal{DE}(\Gamma) \subseteq \mathcal{S}(\Gamma)$*

Proof. The grand coalition and the $n-1$ player coalitions are all members of both $\mathcal{S}(\Gamma)$ and $\mathcal{DE}(\Gamma)$. Let S be a non-saturated coalition with at most $n-2$ players. We will show that S is dually inessential. As S is not saturated there exists $i \in N \setminus S$ such that $c(S) = c(S \cup i)$. Let $S_1 := S \cup i$ and $S_2 := N \setminus i$. Then $S_1 \cup S_2 = N$ and $S_1 \cap S_2 = S$ therefore we can use Definition 10 since

$$\begin{aligned} c(N) &\geq c(N \setminus i), \\ c(S) &\geq c(S) + c(N \setminus i) - c(N), \\ c(S) &\geq c(S_1) + c(S_2) - c(N). \end{aligned}$$

In other words S appears in an overlapping decomposition of S_1 and S_2, therefore it can not be dually essential. □

The above lemma suggests that dually essential coalitions are more useful since they define a smaller characterization set, which in turn implies a smaller LP. However usually it is also harder to determine whether a coalition is dually essential or not. Saturatedness on the other hand can be checked easily. For instance for airport games[1] there exist

[1] This class of games was introduced by Littlechild and Owen (1973).

at most n saturated coalitions, which can be easily determined from the characteristic function. In fact it is enough to know the value of the singleton coalitions to identify the saturated coalitions, which gives us an alternative way to derive an efficient algorithm for the nucleolus.

There is a symmetrical result for essential and dually saturated coalitions.

Lemma 16. *Let $\Gamma = (N, v)$ be a monotonic profit game, then $\mathcal{E}(\Gamma) \subseteq \mathcal{DS}(\Gamma)$.*

Proof. Observe that the singleton coalitions are all members of both $\mathcal{E}(\Gamma)$ and $\mathcal{DS}(\Gamma)$. Let S be a dually non-saturated coalition such that $|S| > 1$. Then there exists $i \in S$ such that $v(S) = v(S \setminus i)$. By monotonicity $v(i) \geq 0$, hence $v(S) \leq v(S \setminus i) + v(i)$. Thus S is inessential. □

When designing a characterisation set we may run into two kinds of difficulties. The collection can be two small to describe (span) every other coalition. A more tricky problem is what we call a *cycle in the decomposition*[2]. This occurs when we try to discard a coalition S using coalition T that was previously excluded *because of* S. For instance the intersection of essential and dually essential coalitions do not always yield a characterization set. It can happen that there is a series of coalitions $S_1, S_2, \ldots, S_k$, where S_ℓ is excluded because of $S_{\ell+1}$ for $\ell = 1, 2, \ldots, k-1$ and $S_k = S_1$. The circular argument leads to a contradiction when we try to use Theorem 6 to verify that the intersection is indeed a characterisation set. A simple example for this is any additive game. Let $\Gamma = (N, v)$ such that $v(S) = \sum_{i \in S} v(i)$. Then only the singleton coalitions and the grand coalition are essential and the $(n-1)$-person coalitions and the grand coalition are dually essential. Thus $\mathcal{E}(\Gamma) \cap \mathcal{DE}(\Gamma) = \{N\}$ and the grand coalition alone does not characterize the nucleolus in any game.

A collection of coalitions $\mathcal{B}_S \subseteq 2^N$ is said to be *S-balanced* if there exist positive weights λ_T, $T \in \mathcal{B}_S$, such that $\sum_{T \in \mathcal{B}_S} \lambda_T e_T = e_S$. An N-balanced collection is simply called *balanced*. A coalition S is called *vital* if for any S-balanced collection $\mathcal{B}_S$ and any system $(\lambda_T)_{T \in \mathcal{B}_S}$ of balancing weights for $\mathcal{B}_S$, $\sum_{T \in \mathcal{B}_S} \lambda_T v(T) < v(S)$. By definition every vital coalition is essential. The concept was introduced by Gillies (1959) and further analyzed by Shellshear and Sudhölter (2009). Huberman (1980) showed that vital coalitions do not necessarily characterize the nucleolus. The next theorem provides a sufficient condition for $\mathcal{E}(\Gamma) \cap \mathcal{DE}(\Gamma)$ to be a characterization set.

Theorem 17. *Let $\Gamma = (N, v)$ a monotonic game with a non-empty core. The collection $\mathcal{E}(\Gamma) \cap \mathcal{DE}(\Gamma)$ forms a characterization set of $\mathcal{N}(\Gamma)$ if the grand coalition is vital.*

[2] Strongly essential coalitions that were introduced in (Brânzei, Solymosi, and Tijs, 2005) are not immune to this kind of failure, hence they do not form a characterization set for the nucleolus.

Proof. If there is no cycle in the decomposition, then by Theorem 6 $\mathcal{E}(\Gamma) \cap \mathcal{DE}(\Gamma)$ is a characterization set. By contradiction suppose that the grand coalition is vital, but there is a cycle $T_1, T_2, \ldots, T_r$ in the decomposition. Note that we may assume without loss of generality that the series alternates between inessentiality and dual inessentiality. If T_ℓ and $T_{\ell+1}$ were deemed redundant for the same reason (e.g. they are both inessential) then the inequality that shows inessentiality of T_ℓ can be refined by the inequality that shows inessentiality of $T_{\ell+1}$. Let us assume that T_1 is inessential – the proof is the same if T_1 is dually inessential. Thus using the definition of essentiality and dual essentiality

$$v(T_1) \leq v(T_2) + \sum_{j=1}^{k_1} v(S_j^1) \tag{1}$$

$$v(T_2) \leq v(T_3) + \sum_{j=1}^{k_2} v(S_j^2) - k_2 \cdot v(N) \tag{2}$$

$$v(T_4) \leq v(T_5) + \sum_{j=1}^{k_3} v(S_j^3) \tag{3}$$

$$\vdots$$

$$v(T_r) \leq v(T_1) + \sum_{j=1}^{k_r} v(S_j^r) - k_r \cdot v(N) \tag{4}$$

In words T_1 is inessential because of the collection $T_2, S_1^1, \ldots, S_{k_1}^1$ (these are all essential coalitions, thus the inequality cannot be refined any more). Then T_2 is dually inessential because of the collection $T_3, S_1^2, \ldots, S_{k_2}^2$ compose an overlapping decomposition of T_2 (and these are all dually essential). And so on until finally T_r is deemed redundant because of $T_1, S_1^r, \ldots, S_{k_r}^r$. Note that there may be coalitions among $S_1^1, \ldots, S_1^2, \ldots, S_1^r, \ldots, S_{k_r}^r$ that coincide. Using indicator functions and the conditions of inessentiality and dual inessentiality.

$$e_{T_1} = e_{T_2} + \sum_{j=1}^{k_1} e_{S_j^1} \tag{5}$$

$$e_{T_2} = e_{T_3} + \sum_{j=1}^{k_2} e_{S_j^2} - k_2 \cdot e_N \tag{6}$$

$$e_{T_4} = e_{T_5} + \sum_{j=1}^{k_3} e_{S_j^3} \tag{7}$$

$$\vdots$$

$$e_{T_r} = e_{T_1} + \sum_{j=1}^{k_r} e_{S_j^r} - k_r \cdot e_N \tag{8}$$

Thus by summing (1)-(4) we obtain that

$$v(N) \leq \frac{1}{k_2 + k_4 + \cdots + k_r} \sum_{i=1}^{r} \sum_{j=1}^{k_i} v(S_j^i)$$

while from (5)-(8) we gather that

$$e_N = \frac{1}{k_2 + k_4 + \cdots + k_r} \sum_{i=1}^{r} \sum_{j=1}^{k_i} e_{S_j^i}$$

i.e. the collection $S_1^1, \ldots, S_1^2, \ldots, S_1^r, \ldots, S_{k_r}^r$ is balanced. This contradicts the fact that the grand coalition is vital. □

4 Verifying the nucleolus

Faigle, Kern, and Kuipers (1998) conjecture the decision problem: "given $x^* \in \mathbb{R}^N$, is x^* the nucleolus?" to be $\mathcal{NP}$-hard in general. The most resourceful tool regarding this problem is the criterion developed by (Kohlberg, 1971).

Theorem 18. *(Kohlberg, 1971) Let $\Gamma = (N, v)$ be a game with non-empty core and let $x \in I(\Gamma)$. Then $x = \mathcal{N}(\Gamma)$ if and only if for all $y \in \mathbb{R}$ the collection $\{\emptyset \neq S \subset N \mid sat_\Gamma(S, x) \leq y\}$ is balanced or empty.*

The Kohlberg-criterion is often used to verify the nucleolus in practical computation when the size of the player set is not too large. As the following LP shows it is easy to tell whether a given collection of coalitions is balanced or not. Let $S_1, \ldots, S_m$ be the collection which balancedness is in question and let $q \in [0, 1]^m$. For $k = 1, \ldots, m$ let

$$\begin{aligned} &p_k^* = \max q_k \\ &\sum_{i=1}^{m} q_i e_{S_i} = e_N \\ &q_1, \ldots, q_m \geq 0. \end{aligned} \tag{9}$$

Lemma 19. *The collection $S_1, \ldots, S_m$ is balanced if and only if $p_k^* > 0$ for each $k = 1, \ldots, m$.*

Proof. Trivially $p_k^* > 0$ is a necessary condition. Let $a_1, \ldots, a_m \in [0, 1]$ be arbitrary reals such that $a_1 + \cdots + a_m = 1$. Furthermore let q^k be an optimal solution of the k^{th} LP. We claim that $\lambda = \sum_{i=1}^m a_i q^i$ is a vector of balancing weights. Note that $\lambda_j = \sum_{i=1}^m a_i q_j^i > 0$ as $q_j^j = p_j^* > 0$ and $\sum_{j=1}^m q_j^i e_{S_j} = e_N$ for all $i = 1, \ldots, m$ by (9). Then

$$\sum_{j=1}^{m} \lambda_j e_{S_j} = \sum_{j=1}^{m} \sum_{i=1}^{m} a_i q_j^i e_{S_j} = \sum_{i=1}^{m} a_i \sum_{j=1}^{m} q_j^i e_{S_j} = (a_1 + \cdots + a_m) e_N = e_N.$$

□

Sobolev (1975) extended Theorem 18 to the prenucleolus (where instead of $x \in I(\Gamma)$ we only require x to be an allocation). A direct consequence of the Sobolev-criterion is that the prenucleolus of monotonic games is non-negative.

Theorem 20. *Let $\Gamma = (N, v)$ be a monotonic game and let z denote its prenucleolus, then $z(i) \geq 0$ for all $i \in N$.*

Proof. By contradiction suppose that $z(i) < 0$ for some $i \in N$. Let $\mathcal{B}_0$ contain the coalitions with the smallest satisfaction values under z. By the Sobolev-criterion $\mathcal{B}_0$ is a balanced collection. For every $S \in \mathcal{B}_0$, $i \in S$ otherwise $S \cup \{i\}$ would have an even smaller satisfaction due to the monotonicity of the characteristic function and the fact that $z(i) < 0$. By balancedness of $\mathcal{B}_0$, $\sum_{S \in \mathcal{B}_0} \lambda_S e_S = e_N$. As $i \in S$ for all $S \in \mathcal{B}_0$ this also means that $\sum_{S \in \mathcal{B}_0} \lambda_S = 1$. Then for all $j \neq i$ and for all $S \in \mathcal{B}_0$, S must contain j. Thus the only coalition in $\mathcal{B}_0$ is the grand coalition. However the grand coalition has zero satisfaction under any allocation, while coalition $\{i\}$ has negative satisfaction under z, which contradicts that $\mathcal{B}_0$ contains the coalitions with the smallest satisfaction values. □

Reijnierse and Potters (1998) proved that for every game (N, v) there exists a collection with at most $2(n-1)$ coalitions that determine the nucleolus. Although finding these coalitions is as hard as computing the nucleolus itself. Unfortunately this result does not make the verification of the nucleolus belong to $\mathcal{NP}$. Even if somehow we could effortlessly put the satisfaction values of the 2^n coalition in increasing order. It can happen that these $2(n-1)$ coalitions are scattered among the different balanced coalition arrays and we have to evaluate exponential many of them before we could confirm that the given allocation is indeed the nucleolus.

The Kohlberg-criterion applied to games with coalition formation restrictions yields the following theorem.

Theorem 21. *Let $\mathcal{F}$ be a characterization set and x be an imputation of the game Γ with $\mathcal{C}(\Gamma) \neq \emptyset$. Then $x = \mathcal{N}(\Gamma)$ if and only if for all $y \in \mathbb{R}$ the collection $\{S \in \mathcal{F} \mid sat_\Gamma(S, x) \leq y\}$ is balanced or empty.*

For a proof see (Maschler, Potters, and Tijs, 1992). A similar criterion appears in (Groote Schaarsberg, Borm, Hamers, and Reijnierse, 2013). With the help of the Kohlberg-criterion the problem of finding the nucleolus is reduced to finding the right characterization set.

5 The nucleolus of bankruptcy games

The game theoretic analysis of bankruptcy games was initiated by O'Neill (1982), but Aumann and Maschler (1985) made the problem really popular[3] by proving the equivalence of the Talmud-rule and the nucleolus. Although their result was spectacular the proof used complicated concepts like the reduced game or kernel. Many believed that an elementary proof should exists for this problem. Benoît (1997) was the first to publish a simplification, although he still needed long pages of computation to reach the desired result. Recently Fleiner and Sziklai (2012) managed to provide an elementary and instructive proof with the help of the hydraulic framework that was developed by Kaminski (2000).

The differences between the approaches are remarkable. The hydraulic proof seems to be straightforward enough but the elapsed time between the original and this proof signals that a few subtle tricks were needed to overcome the difficulties. It seems that for each particular game class a different idea is needed to compute the nucleolus. Bankruptcy games make a textbook examples why characterization sets are so resourceful. The benefits of Theorem 6, 11 and 17 all come together. Here – after defining bankruptcy games – we disclose a simple treatment of the problem, which is barely two-page long.

Let $N = \{1, 2, \ldots, n\}$ be the set of creditors. The *bankruptcy problem*[4] is defined as a pair (d, E) where $E \in \mathbb{R}_+$ represents the firm's liquidation value (or *estate/endowment*) and $d \in \mathbb{R}^n_+$ is the collection of claims with $\sum_{i=1}^n d_i > E$. A *solution of a bankruptcy problem* is a vector $x \in \mathbb{R}^n_+$ such that $\sum_i^n x_i = E$.

The characteristic function corresponding to the bankruptcy problem (d, E) is

$$v_{(d,E)}(S) = \max\{E - d(N \setminus S), 0\}$$

By definition, this is the value that is left from the firm's liquidation value $E = v_{(d,E)}(N)$ after the claim of each player of the complement coalition $N \setminus S$ has been satisfied. Coalition S can achieve $v_{(d,E)}(S)$ without any effort. Note that $v_{(d,E)}$ is non-negative and supermodular. Hence bankruptcy games are convex, which implies that they are superadditive and monotonic as well.

The hydraulic proof of Fleiner and Sziklai (2012) hints that only the singleton coalitions and the $n - 1$ person coalitions are relevant in the computation of the nucleolus. Let us investigate whether any of the characterization sets coincide with this collection of coalitions.

[3]After Aumann and Maschler's seminal paper the literature virtually exploded. A recent comprehensive survey of Thomson (2015) lists nearly 200 references!

[4]Sometime it is referred as the *claims* or *rationing problem*.

Lemma 22. *Let (d, E) be a bankruptcy problem and $\Gamma = (N, v_{(d,E)})$ be the corresponding bankruptcy game. Then $\mathcal{E}(\Gamma)$ is a subset of the singleton coalitions, the grand coalition and coalitions with non-zero characteristic function value.*

Proof. Let $S \subset N$ be such that $|S| > 1$ and $v_{(d,E)}(S) = 0$. It follows from monotonicity of $v_{(d,E)}$ that $v_{(d,E)}(T) = 0$ for any $T \subset S$. Thus for any partition $S_1, \ldots, S_k$ of S,

$$v_{(d,E)}(S) \leq v_{(d,E)}(S_1) + \cdots + v_{(d,E)}(S_k).$$

That is, S is inessential. □

In fact $\mathcal{E}(\Gamma)$ contains exactly the singleton coalitions, the grand coalition and coalitions with non-zero characteristic function value. We will come back to this question later, since we do not need it for the computation of the nucleolus.

Lemma 23. *Let (d, E) be a bankruptcy problem and $\Gamma = (N, v_{(d,E)})$ be the corresponding bankruptcy game. Then $\mathcal{DE}(\Gamma)$ is a subset of the $n-1$ player coalitions, the grand coalition and coalitions with characteristic function value of zero.*

Proof. By default $\mathcal{DE}(\Gamma)$ contains all the $n-1$ player coalitions. Let $S \subset N$ be such that $|S| < n-1$ and $v_{(d,E)}(S) > 0$. It follows from monotonicity of $v_{(d,E)}$ that $v_{(d,E)}(T) > 0$ for any $S \subset T$. Thus for any overlapping decomposition $T_1, \ldots, T_k$ of S,

$$\begin{aligned} E - d(N \setminus S) &\leq E - d(N \setminus S) \\ E - d(N \setminus S) &\leq kE - d(N \setminus S) - (k-1)E \\ E - d(N \setminus S) &\leq E - d(N \setminus T_1) + \cdots + E - d(N \setminus T_k) - (k-1)v_{(d,E)}(N) \\ v_{(d,E)}(S) &\leq v_{(d,E)}(T_1) + \cdots + v_{(d,E)}(T_k) - (k-1)v_{(d,E)}(N) \end{aligned}$$

where we used that $d(N \setminus S) = d(N \setminus T_1) + \cdots + d(N \setminus T_k)$. Thus S is dually inessential. □

It seem that neither of these two characterization sets coincide with the desired collection. A natural idea is to examine the intersection of these two sets.

Observation 24. *Let (d, E) be a bankruptcy problem and $\Gamma = (N, v_{(d,E)})$ be the corresponding bankruptcy game. Then $\mathcal{E}(\Gamma) \cap \mathcal{DE}(\Gamma)$ is subset of the grand coalition, $n-1$ person coalitions with non-zero characteristic function value and singleton coalitions with characteristic function value of zero.*

Although Observation 24 gives us the collection of coalitions that we have looked for, we still need to prove that $\mathcal{E}(\Gamma) \cap \mathcal{DE}(\Gamma)$ is actually a characterization set itself. For this we will show that the grand coalition is vital.

Theorem 25. *Let (d, E) be a bankruptcy problem and $\Gamma = (N, v_{(d,E)})$ be the corresponding bankruptcy game. Then*

$$\{i \in N \mid v_{(d,E)}(i) = 0\} \cup \{N \setminus i \mid i \in N,\ v_{(d,E)}(N \setminus i) > 0\} \cup \{N\}$$

is a characterization set for $\mathcal{N}(\Gamma)$.

Proof. Due to Theorem 17 we only need to prove that the grand coalition is vital. Suppose by contradiction that the grand coalition is not vital, that is, there exists a collection $\mathcal{B} \subset 2^N$ and positive balancing weights $\{\lambda_T > 0 \mid T \in \mathcal{B}\}$ such that $\sum_{T\in\mathcal{B}} \lambda_T e_T = e_N$ and

$$v_{(d,E)}(N) \leq \sum_{T\in\mathcal{B}} \lambda_T v_{(d,E)}(T). \tag{10}$$

Let $\mathcal{B}_+ \subseteq \mathcal{B}$ denote those coalitions for which $v_{(d,E)}(T)$ is not zero, then Eq. (10) can be written as

$$v_{(d,E)}(N) \leq \sum_{T\in\mathcal{B}_+} \lambda_T v_{(d,E)}(T) = \sum_{T\in\mathcal{B}_+} \lambda_T (E - d(N \setminus T)).$$

Note that $\sum_{T\in\mathcal{B}_+} \lambda_T > 1$ otherwise $v_{(d,E)}(N) = E > \sum_{T\in\mathcal{B}_+} \lambda_T (E - d(N \setminus T))$. Then

$$v_{(d,E)}(N) \leq \sum_{T\in\mathcal{B}_+} \lambda_T (E - d(N \setminus T)) = \sum_{T\in\mathcal{B}_+} \lambda_T (E - d(N) + d(T)) =$$
$$\sum_{T\in\mathcal{B}_+} \lambda_T (E - d(N)) + \sum_{T\in\mathcal{B}_+} \lambda_T d(T) \leq \sum_{T\in\mathcal{B}_+} \lambda_T (E - d(N)) + \sum_{T\in\mathcal{B}} \lambda_T d(T).$$

Using that $\sum_{T\in\mathcal{B}} \lambda_T d(T) = d(N)$ we obtain

$$E - d(N) \leq \sum_{T\in\mathcal{B}_+} \lambda_T (E - d(N)),$$
$$1 \geq \sum_{T\in\mathcal{B}_+} \lambda_T,$$

which contradicts that $\sum_{T\in\mathcal{B}_+} \lambda_T > 1$.

Since the grand coalition is vital $\mathcal{E}(\Gamma) \cap \mathcal{DE}(\Gamma)$ is a characterization set. From Lemma 22 and 23 it follows that

$$\mathcal{E}(\Gamma) \cap \mathcal{DE}(\Gamma) \subseteq \{i \in N \mid v_{(d,E)}(i) = 0\} \cup \{N \setminus i \mid i \in N,\ v_{(d,E)}(N \setminus i) > 0\} \cup \{N\}.$$

Any enlargement of a characterization set is a characterization set by Corollary 7, thus we are done. ◻

We obtained a collection – whose size is linear in the number of players – that characterize the nucleolus in every bankruptcy game. For sake of completeness we provide the couterparts of Lemma 22 and 23.

Theorem 26. *Let (d, E) be a bankruptcy problem and $\Gamma = (N, v_{(d,E)})$ be the corresponding bankruptcy game. Then*

a. *$\mathcal{E}(\Gamma)$ contains the singleton coalitions, the grand coalition and coalitions with non-zero characteristic function value,*

b. *$\mathcal{DE}(\Gamma)$ contains the $n-1$ player coalitions, the grand coalition and coalitions with characteristic function value of zero.*

Proof. In light of Lemma 22 it is enough to prove that coalitions with non-zero characteristic function value are essential. By contradiction suppose that $v_{(d,E)}(S) > 0$ and S is inessential. Then there exists a partition $T_1, \ldots, T_{k+1}$ of S such that $v_{(d,E)}(S) \leq \sum_{i=1}^{k+1} v_{(d,E)}(T_i)$. Some of the $v_{(d,E)}(T_i)$ values may be zeros. By uniting these coalitions the characteristic function may only increase. Thus we may assume that $v(T_i) > 0$ for $i = 1, \ldots, k$ and $v_{(d,E)}(T_{k+1}) = 0$ where we allow T_{k+1} to be the empty set. Notice that $k \geq 2$ in this setting.

$$v_{(d,E)}(S) \leq \sum_{i=1}^{k+1} v_{(d,E)}(T_i) = \sum_{i=1}^{k} v_{(d,E)}(T_i),$$
$$E - d(N \setminus S) \leq \sum_{i=1}^{k} (E - d(N \setminus T_i)) = k(E - d(N)) + \sum_{i=1}^{k} d(T_i).$$

By subtracting $\sum_{i=1}^{k} c(T_i)$ from both sides and estimating the sum from below we get

$$E - d(N) \leq E - d(N \setminus T_{k+1}) = E - d(N \setminus S) - \sum_{i=1}^{k} d(T_i) \leq k(E - d(N)),$$
$$1 \geq k.$$

which contradicts that $k \geq 2$.

For the second part of the Theorem we need prove that coalitions with characteristic function value of zero are dually essential. By contradiction suppose that $v_{(d,E)}(S) = \max\{E - d(N \setminus S), 0\} = 0$ and S is dually inessential. By perturbing the claims with a small positive number we can always achieve that no collection of claims sum up to the estate, therefore we may also suppose that $E - d(N \setminus S) < 0$. Then there exists an overlapping decomposition $T_1, \ldots, T_k, T_{k+1}, \ldots, T_\ell$ of S such that

$$v_{(d,E)}(S) \leq \sum_{i=1}^{\ell} v_{(d,E)}(T_i) - (\ell - 1)v_{(d,E)}(N).$$

Some of the $v_{(d,E)}(T_i)$ values may be zeros. We may assume that $v(T_i) > 0$ for $i = 1, \ldots, k$ and $v_{(d,E)}(T_i) = 0$ for $i = k+1, \ldots, \ell$ where we allow $k = \ell$.

$$0 = v_{(d,E)}(S) \leq \sum_{i=1}^{\ell} v_{(d,E)}(T_i) - (\ell - 1)v_{(d,E)}(N) = \sum_{i=1}^{k} v_{(d,E)}(T_i) - (\ell - 1)v_{(d,E)}(N) =$$

$$k(E - d(N)) + \sum_{i=1}^{k} d(T_i) - (\ell - 1)E \leq k(E - d(N)) + (k-1)d(N) + d(S) - (\ell - 1)E \leq$$

$$kE - d(N) + d(S) - (k-1)E = E - d(N \setminus S) < 0.$$

which is clearly a contradiction. Note that we used that

$$\sum_{i=1}^{k} d(T_i) \leq (k-1)d(N) + d(S),$$

which follows from the fact that $T_1, \ldots, T_k, T_{k+1}, \ldots, T_\ell$ compose an overlapping decomposition of S. □

In view of Theorem 26 we can easily construct a bankruptcy game where every coalition is dually essential or a game where every coalition is essential. What is more we can define a game Γ with n players where both the size of $\mathcal{E}(\Gamma)$ and $\mathcal{DE}(\Gamma)$ is $\mathcal{O}(2^n)$. Instead of proving that the grand coalition is vital we could simply derive Theorem 25 from Theorem 6. This method is also instructive since it shreds some light on the structure of satisfaction values, thus we present it as well.

Second proof. We need to prove that when we exclude coalition S because it is (dually) inessential, then there is an (overlapping decomposition) partition of S, containing coalitions that belong to $\mathcal{E}(\Gamma) \cap \mathcal{DE}(\Gamma)$ and have smaller satisfaction values than S. There are two cases. If $v_{(d,E)}(S) = 0$, then by monotonicity $v_{(d,E)}(i) = 0$ for all $i \in S$. Thus $sat_\Gamma(S, x) \geq sat_\Gamma(i, x)$ for all $i \in S$ and for any core allocation x. Furthermore $i \in \mathcal{E}(\Gamma) \cap \mathcal{DE}(\Gamma)$ for all $i \in S$ and naturally e_S can be expressed as a linear combination of $\{e_i :\ i \in S\}$. If $v_{(d,E)}(S) > 0$, then by monotonicity $v_{(d,E)}(N \setminus i) > 0$ for all $i \in N \setminus S$. The collection $\{N \setminus i \mid i \in N \setminus S\}$ is an overlapping decomposition of S, furthermore for any core allocation x (cf. Lemma 23)

$$v_{(d,E)}(S) \leq \sum_{i \in N \setminus S} v_{(d,E)}(N \setminus i) - (|N \setminus S| - 1) v_{(d,E)}(N)$$

$$x(S) - v_{(d,E)}(S) \geq -\sum_{i \in N \setminus S} v_{(d,E)}(N \setminus i) + (|N \setminus S| - 1) x(N) + x(S)$$

$$x(S) - v_{(d,E)}(S) \geq \sum_{i \in N \setminus S} x(N \setminus i) - \sum_{i \in N \setminus S} v_{(d,E)}(N \setminus i)$$

$$sat_\Gamma(S, x) \geq \sum_{i \in S} sat_\Gamma(N \setminus i, x)$$

Thus $sat_\Gamma(S, x) \geq sat_\Gamma(N \setminus i, x)$ for all $i \in N \setminus S$. Again $N \setminus i \in \mathcal{E}(\Gamma) \cap \mathcal{DE}(\Gamma)$ for all $i \in N \setminus S$ and e_S can be expressed as a linear combination of $\{e_{N \setminus i} : i \in N \setminus S\}$ and e_N. By Theorem 6 we conclude that $\mathcal{E}(\Gamma) \cap \mathcal{DE}(\Gamma)$ is a characterization set for the nucleolus of Γ. □

6 Literature overview

There are many ingenious techniques to compute the nucleolus of various classes of cooperative games. A standard method is to verify the nucleolus through the satisfaction vector, i.e. guessing what the solution is, then checking whether the satisfaction vector of the proposed allocation is lexicographically maximal. For example such a method is used to find the nucleolus of voting games with a non-empty core (Elkind, Goldberg, Goldberg, and Wooldridge, 2009) and the nucleolus of standard tree games (Megiddo, 1978).

A more advanced technique of this kind when the balancedness of the satisfaction vector is examined. The Kohlberg-criterion is mostly used in combination with other methods as proving balancedness for abstract coalition structures can be a challenging task. In comparison verifying balancedness of a given collection of coalitions can be done with a simple LP (cf. Lemma 19). The Chinese postman game that was introduced by Granot, Hamers, Kuipers, and Maschler (2011) provides an example where balancedness is crucial in the proof. The effectiveness of the painting algorithm of Maschler, Potters, and Reijnierse (2010) is also proven with the help of Kohlberg-criterion.

The application of the axiomatization of the nucleolus is more intricate. It is usually hard to confirm whether a solution admits the reduced game property. Only when the structure of the game is simple enough, that is, the reduced game falls into the same game class, it is possible to use the axiomatization[5]. Brânzei, Iñarra, Tijs, and Zarzuelo (2006) used this technique the find the nucleolus of airport profit games. Another perhaps more famous example that uses reduced game property is the Theorem of Aumann and Maschler (1985) which states that the Talmud-rule yields the nucleolus in case of bankruptcy games.

[5]For instance the reduced game of a standard tree game is not a standard tree game.

The linear programming approach received substantial attention. The sequential LP of Maschler, Peleg, and Shapley (1979) was the first that was suitable for computational purposes. In general this method needs $\mathcal{O}(4^n)$ number of linear programs with constraint coefficients in $\{-1, 0, 1\}$. Since then there have been many attempts to improve the computation process either by restraining the number of LPs or by finding a unique minimization problem with minimal number of constraints. Sankaran (1991) provided a method that needs $\mathcal{O}(2^n)$ number of LPs with constraint coefficients in $\{-1, 0, 1\}$. Later Fromen (1997) improved his results. Potters, Reijnierse, and Ansing (1996) proposed a fast algorithm to find the nucleolus of any game with non-empty imputation set. This algorithm is based on solving a prolonged simplex algorithm. It requires solving $n - 1$ linear programs with $\mathcal{O}(2^n)$ number of rows and columns. The most recent result is by Puerto and Perea (2013). They offered a unique minimization problem with $\mathcal{O}(4^n)$ constraints where the coefficients are from the set $\{-1, 0, 1\}$. An interesting addition to this topic is provided by Guajardo and Jörnsten (2014). They collect examples from the literature where the nucleolus was miscalculated and analyze what went wrong.

By itself the linear programming approach is not an effective tool as we either need a sequential LP with exponential many programs or a unique maximization problem with exponential many constraints. The LP approach is often used to calculate the nucleolus in practice when the number of players is limited and no theory is available. There are quite a few instances when the opposite is true: an LP helps to derive a theoretical result, for an example see (Kamiyama, 2014) or (Kern and Paulusma, 2003).

Surprisingly few papers use the concept of characterization-sets explicitly – the main theoretical advancement that was developed parallel by Granot, Granot, and Zhu (1998) and Reijnierse and Potters (1998). Some papers like (Kamiyama, 2014) and (Brânzei, Solymosi, and Tijs, 2005) exploit this idea but there are many others which use it unknowingly. For instance Maschler, Potters, and Reijnierse (2010) identify a collection that determines the core and nucleolus of standard tree games which is in fact a characterization set. However they do not make the connection between their method and the above mentioned two papers.

The primary contribution of this paper is the expansion of the theory of characterization sets. When the game in question is well-structured, characterization sets can simplify the proof substantially. Even when the structure is more complicated characterization sets can make the proof significantly simpler or at least possible.

Naturally characterization sets do not make other approaches obsolete. The algorithm for the nucleolus of bankruptcy games uncovered by Aumann and Maschler (1985) is considerably faster than the LP approach. Also from didactical point of view the hydraulic representation of Kaminski (2000) is much more instructive. The main advantage of characterization sets is their algebraic formalism. The proofs need less bag-tricks and more mechanical computation which is perhaps aesthetically less pleasing but much more effective in terms of results.

Finally let us stress that the four universal characterization sets that we analyzed in this paper can only help when the game in question has a non-empty core. Göthe-Lundgren, Jörnsten, and Värbrand (1996) computed the nucleolus of a vehicle routing game using the concept of essential coalitions. However as Chardaire (2001) pointed out these games are not necessarily balanced, hence their approach is flawed.

Acknowledgements

The authors would like to thank Prof. Tamás Kis of Institute for Computer Science and Control (MTA SZTAKI) for his valuable comments. Research was funded by OTKA grants K101224 and K108383 and by the Hungarian Academy of Sciences under its Momentum Programme (LD-004/2010).

References

Arin, J., and E. Inarra (1998): "A Characterization of the Nucleolus for Convex Games," *Games and Economic Behaviour*, 23, 12–24.

Aumann, R., and M. Maschler (1985): "Game theoretic analysis of a bankruptcy problem from the Talmud," *Journal of Economic Theory*, 36, 195–213.

Benoît, J.-P. (1997): "The nucleolus is contested-garment-consistent: a direct proof," *Journal of Economic Theory*, 77, 192–196.

Brânzei, R., E. Iñarra, S. Tijs, and J. Zarzuelo (2006): "A simple algorithm for the nucleolus of airport profit games," *International Journal of Game Theory*, 34, 259–272.

Brânzei, R., T. Solymosi, and S. Tijs (2005): "Strongly essential coalitions and the nucleolus of peer group games," *International Journal of Game Theory*, 33, 447–460.

Chardaire, P. (2001): "The core and nucleolus of games: A note on a paper by Goethe-Lundgren et al.," *Mathematical Programming*, 90(1).

Deng, X., Q. Fang, and X. Sun (2009): "Finding nucleolus of flow game," *Journal of Combinatorial Optimization*, 18(1), 64–86.

Elkind, E., L. A. Goldberg, P. Goldberg, and M. Wooldridge (2009): "Computational complexity of weighted voting games," *Annals of Mathematics and Artificial Intelligence*, 56, 109–131.

Faigle, U., W. Kern, and J. Kuipers (1998): "Computing the Nucleolus of Min-cost Spanning Tree Games is NP-hard," *International Journal of Game Theory*, 27.

Fleiner, T., and B. Sziklai (2012): "The Nucleolus Of The Bankruptcy Problem By Hydraulic Rationing," *International Game Theory Review*, 14(01), 1250007–1–1.

Fromen, B. (1997): "Reducing the number of linear programs needed for solving th nucleolus problem of n-person game," *European Journal of Operntional Research*, 98, 626–636.

Gillies, D. (1959): "Solutions to general non-zero-sum games," in *Contributions to the Theory of Games IV.*, ed. by A. Tucker, and R. E. Luce, vol. 40 of *Ann. Math. Stud.*, pp. 47–85. Princeton University Press.

Göthe-Lundgren, M., K. Jörnsten, and P. Värbrand (1996): "On the nucleolus of the basic vehicle routing game," *Mathematical Programming*, 72, 83–100.

Granot, D., F. Granot, and W. R. Zhu (1998): "Characterization sets for the nucleolus," *International Journal of Game Theory*, 27(3), 359–374.

Granot, D., H. Hamers, J. Kuipers, and M. Maschler (2011): "On Chinese postman games where residents of each road pay the cost of their road," *Games and Economic Behavior*, 72, 427–438.

Groote Schaarsberg, M., P. Borm, H. Hamers, and H. Reijnierse (2013): "Game theoretic analysis of maximum cooperative purchasing situations," *Naval Research Logistics*, 60, 607–624.

Guajardo, M., and K. Jörnsten (2014): "Common mistakes in computing the nucleolus," Discussion Paper 2014/15, Norwegian School of Economics.

Huberman, G. (1980): "The nucleolus and essential coalitions," in *Analysis and Optimization of Systems*, ed. by A. Bensoussan, and J. L. Lions, vol. 28 of *Lecture Notes in Control and Information Sciences*, pp. 416–422. Elsevier B.V.

Kaminski, M. (2000): "'Hydraulic' rationing," *Mathematical Social Sciences*, 40(2), 131–155.

Kamiyama, N. (2014): "The nucleolus of arborescence games in directed acyclic graphs," *Operations Research Letters*, forthcoming.

Kern, W., and D. Paulusma (2003): "Matching games: the least core and the nucleolus," *Mathematics of Operations Research*, 28(2), 294–308.

Kohlberg, E. (1971): "On the nucleolus of a characteristic function game," *SIAM Journal on Applied Mathematics*, 20, 62–65.

Kuipers, J. (1996): "A polynomial time algorithm for computing the nucleolus of convex games," Report m 96-12, University of Maastricht.

Littlechild, S. C., and G. Owen (1973): "A Simple Expression for the Shapley Value in a Special Case," *Management Science*, 20(3), 370–372.

Maschler, M., B. Peleg, and L. Shapley (1979): "Geometric properties of the kernel, nucleolus and related solution concepts," *Math. Oper. Res.*, 4, 303–338.

Maschler, M., J. Potters, and H. Reijnierse (2010): "The nucleolus of a standard tree game revisited: a study of its monotonicity and computational properties," *International Journal of Game Theory*, 39(1-2), 89–104.

Maschler, M., J. Potters, and S. Tijs (1992): "The General Nucleolus and the Reduced Game Property," *International Journal of Game Theory*, 21, 85–106.

Megiddo, N. (1974): "Nucleoluses of compound simple games," *SIAM Joournal of Applied Mathematics*, 26(3).

——— (1978): "Computational Complexity of the Game Theory Approach to Cost Allocation for a Tree," *Mathematics of Operations Research*, 3(3), 189–196.

O'Neill, B. (1982): "A problem of rights arbitration from the Talmud," *Mathematical Social Sciences*, 2, 345–371.

Potters, J. A. M., H. Reijnierse, and M. Ansing (1996): "Computing the Nucleolus by Solving a Prolonged Simplex Algorithm," *Mathematics of Operations Research*, 21(3), 757–768.

Puerto, J., and F. Perea (2013): "Finding the nucleolus of any n-person cooperative game by a single linear program," *Computers & OperationsResearch*, 40, 2308–2313.

Reijnierse, H., and J. A. M. Potters (1998): "The $\mathcal{B}$-nucleolus of TU-games," *Games and Economic Behavior*, 24, 77–96.

Sankaran, J. K. (1991): "On finding the nucleolus of an n-person cooperative game," *International Journal of Game Theory*, 19, 329–338.

Schmeidler, D. (1969): "The Nucleolus of a Characteristic Function Game," *SIAM Journal on Applied Mathematics*, 17, 1163–1170.

Shellshear, E., and P. Sudhölter (2009): “On core stability, vital coalitions, and extendability,” *Games and Economic Behavior*, 67(2), 633–644.

Sobolev, A. (1975): “A characterization of optimality principles in cooperative games by functional equations (Russian),” *Math. Methods Soc. Sci.*, 6, 94–151.

Solymosi, T., and T. E. S. Raghavan (1994): “An Algorithm for Finding the Nucleolus of Assignment Games,” *International Journal of Game Theory*, 23, 119–143.

Thomson, W. (2015): “Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: An update,” *Mathematical Social Sciences*, forthcoming.