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**Institute of Economics, Research Centre for Economic and Regional Studies,  
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## Abstract

A common feature of the Hungarian, Irish, Spanish and Turkish higher education admission systems is that the students apply for programmes and they are ranked according to their scores. Students who apply for a programme with the same score are in a tie. Ties are broken by lottery in Ireland, by objective factors in Turkey (such as date of birth) and other precisely defined rules in Spain. In Hungary, however, an equal treatment policy is used, students applying for a programme with the same score are all accepted or rejected together. In such a situation there is only one question to decide, whether or not to admit the last group of applicants with the same score who are at the boundary of the quota. Both concepts can be described in terms of stable score-limits. The strict rejection of the last group with whom a quota would be violated corresponds to the concept of H-stable (i.e. higher-stable) score-limits that is currently used in Hungary. We call the other solutions based on the less strict admission policy as L-stable (i.e. lower-stable) score-limits. We show that the natural extensions of the Gale-Shapley algorithms produce stable score-limits, moreover, the applicant-oriented versions result in the lowest score-limits (thus optimal for students) and the college-oriented versions result in the highest score-limits with regard to each concept. When comparing the applicant-optimal H-stable and L-stable score-limits we prove that the former limits are always higher for every college. Furthermore, these two solutions provide upper and lower bounds for any solution arising from a tie-breaking strategy. Finally we show that both the H-stable and the L-stable applicant-proposing score-limit algorithms are manipulable.

Keywords: college admissions, stable matching, mechanism design

JEL classification: C78, I21

# Egyetemi felvételi stabil ponthatárokkal

Biró Péter - Sofya Kiselgof

## Összefoglaló

A magyar, az ír, a spanyol és a török felsőoktatási felvételik közös vonása, hogy a diákok szakokra jelentkeznek, és mindenhol pontszámaik alapján rangsorolják a jelentkezőket. Ha két diák pontosan ugyanakkora pontszámot ér el egy adott szakon, akkor azt mondjuk, hogy holtversenyben vannak. A holtversenyeket Írországból sorsolással, Törökországban a születési dátum szerint, Spanyolországban pedig egyéb finomított pontozási módszerrel döntenek el. Magyarországon viszont az egyenlő elbánás elve érvényesül, a holtversenyben lévő diákokat vagy mind felveszik, vagy mind elutasítják. Ebben az esetben csak az a kérdés, hogy milyen döntés szülessen olyan holtversenyben lévő diákokról, akik felvételével az adott szak kvótája éppen sérülne. Mindkét lehetséges eset leírható a stabil ponthatárok modelljével. A kvóták szigorú betartását biztosító elv – amely esetén az utolsó csoport holtversenyben lévő diákokat mind elutasítják – az úgynevezett H-stabil ponthatárokkal írható le. Ezt a koncepciót használják Magyarországon is. A másik lehetőséghez tartozó megengedőbb eljárást – amelyben a kvóták csak puha korlátokat jelentenek – L-stabil ponthatárokkal írhatjuk le. Megmutatjuk, hogy a Gale–Shapley-algoritmus természetes általánosításai stabil ponthatárokhoz vezetnek, sőt, a diákok felől futtatott verzió a lehető legalacsonyabb pontszámokat, míg az egyetemek felől futtatott verzió a lehető legmagasabb pontszámokat eredményezi mindkét koncepció szerint. Továbbá, a H-stabil diákoptimális ponthatárok legalább olyan magasak, mint az L-stabil diákoptimális ponthatárok, és bármelyik holtverseny felbontásával kapott diákoptimális megoldás a két fenti megoldás közé esik. Végül megmutatjuk, hogy mind a H-stabil, mind az L-stabil diákok felől futtatott eljárás manipulálható.

Tárgyszavak: egyetemi felvételi, stabil párosítás, mechanizmustervezés

JEL kódok: C78, I21

# College admissions with stable score-limits

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## Abstract

A common feature of the Hungarian, Irish, Spanish and Turkish higher education admission systems is that the students apply for programmes and they are ranked according to their scores. Students who apply for a programme with the same score are in a tie. Ties are broken by lottery in Ireland, by objective factors in Turkey (such as date of birth) and other precisely defined rules in Spain. In Hungary, however, an equal treatment policy is used, students applying for a programme with the same score are all accepted or rejected together. In such a situation there is only one question to decide, whether or not to admit the last group of applicants with the same score who are at the boundary of the quota. Both concepts can be described in terms of *stable score-limits*. The strict rejection of the last group with whom a quota would be violated corresponds to the concept of H-stable (i.e. higher-stable) score-limits that is currently used in Hungary. We call the other solutions based on the less strict admission policy as L-stable (i.e. lower-stable) score-limits. We show that the natural extensions of the Gale-Shapley algorithms produce stable score-limits, moreover, the applicant-oriented versions result in the lowest score-limits (thus optimal for students) and the college-oriented versions result in the highest score-limits with regard to each concept. When comparing the applicant-optimal H-stable and L-stable score-limits we prove that the former limits are always higher for every college. Furthermore, these two solutions provide upper and lower bounds for any solution arising from a tie-breaking strategy. Finally we show that both the H-stable and the L-stable applicant-proposing score-limit algorithms are manipulable.

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## 1 Introduction

Gale and Shapley [14] introduced a model and solution concept to solve the college admissions problem fifty years ago<sup>1</sup>. In their model they suppose that the students submit

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<sup>1</sup>The 2012 Nobel-Prize in Economic Sciences has been awarded to Alvin Roth and Lloyd Shapley for the theory of stable allocations and the practice of market design.

preference lists containing the colleges they apply to, and each college ranks their applicants in a strict order and also provides an upper quota. Based on the submitted preferences a central body computes a fair solution. The fairness criterion they proposed is *stability*, which essentially means that if an application is rejected then it must be the case that the college must have filled its quota with applicants better than the our applicant's concerned. They gave an efficient method to find a stable matching and they proved that it is actually optimal for the students in that sense that no student can be admitted to a better college in another stable matching. The Gale-Shapley algorithm has linear time implementation (see e.g. Knuth ), which means that the running time of the algorithm is proportional to the number of applications. Another attractive property of this matching mechanism, proved by Roth, that it is strategyproof for the students, i.e., no student can be admitted to any better college by submitting false preferences.

Later, it turned out (Roth [22]) that the algorithm proposed by Gale and Shapley had already been implemented in 1952 in the National Resident Matching Program and has been used since to coordinate junior doctor recruitment in the US. Moreover, the very same method has been implemented recently in the Boston [4] and New York [3] high school matching programs. However, college admissions are still organized in a completely decentralized way in the US, with all its flaws, that is unraveling through early admissions and the coordination problems caused by too many or not enough students admitted. See some representative stories on American college admissions practices in the blog of Al Roth [33].

There are many other countries where higher education admissions are more regulated, but yet not centralized. In Russia, the common timetable of the admissions prevent the unraveling and the use of 'original documents' provide better coordination regarding the number of students admitted, but yet the solution is far from being optimal.<sup>2</sup> In the UK, there is a common platform to manage the admissions by UCAS [34] but there is no centralized matching mechanism, the decisions and actions of the users (students and higher education institutions) are still decentralized.

Finally, there are some countries which do have centralized matching schemes for higher education admissions. In particular, there are scientific papers on the Chinese [26, 27], German [10, 25, 29], Hungarian [6, 7], Spanish [20], Turkish [5] schemes.<sup>3</sup>

The Chinese higher education admissions system is certainly the largest in the world, with more that 20 million students enrolled in 2009 [27]. The system is based on a centralized exam, called National College Entrance Examinations, which provides a score assigned to each students and this induce a ranking of the students by universities. The matching process (see [26]) is a kind of Boston-mechanism with some extra tweaks that makes the system manipulable and controversial. The German clearinghouse for higher education admissions deals only with a small segment of subjects (about 13,000 student from the total 500,000, see [29]). The clearinghouse is a mixed system, in the first phase the Boston-mechanism is used and in the second phase the college-proposing Gale-Shapley, so the process is not incentive compatible [10, 25].

The Hungarian, Irish, Spanish and Turkish higher education matching schemes are all

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<sup>2</sup>Each applicant applies to at most five universities, but does not inform universities about her preferences among them. Universities rank students using results of Unified State Exams. Two 'admission rounds' are organized that are similar to the first two steps of a deferred acceptance procedure. After the second step, universities that still have empty seats are allowed to organize additional admissions.

<sup>3</sup>However, we shall note that regrettably these scientific papers deal only with some special features of these systems (as we also do in this paper) so not all the aspects of these schemes are described. But luckily, there is a new European research network, called Matching in Practice [32], one of whose aim is to collect and describe current matching practices in Europe. So hopefully we will have a better picture and understanding on the current practices, at least in Europe.

based on a centralized scoring system. The Irish system has not been described yet in a scientific paper to the best of our knowledge.<sup>4</sup> In the other three countries students are assigned a score with regard to each programme they applied to, these scores are coming mainly from their grades and entrance exams. The scores of a student may differ at two programmes, since when calculating the score of a student for a particular programme only those subjects are considered which are relevant for that programme. The solution of the admission processes are represented by the so-called *score-limits*, which are referred to as 'base scores' in Turkey [5] and 'cutoff marks' in Spain. The score-limit of a programme means the lowest score that allows a student to be admitted to that programme. The score-limits together with the preferences of the students naturally induce a matching, where each student is admitted to the first place on her list where she achieved the score-limit.

In Turkey [5] the ties are broken according to the date of birth of the students and the college-proposing Gale-Shapley algorithm is used. In Spain the scoring method is fine enough (the admission marks are from 5 to 14 with 3 decimal fractions, and some further priority rules are also used), so ties are very unlikely. They use the applicant-proposing Gale-Shapley algorithm with the special feature of limiting the length of the preference lists, a setting that creates strategic issues that were studied in detail by Romero-Medina [20] and Calsamiglia et al. [11].

In fact, in most applications where ties may occur, the programme coordinators break these ties. In the high school matching schemes in New York [3] and Boston [4] lottery is used for breaking ties. However, this may lead to suboptimal solutions as Erdil and Erkin [12] pointed out, but according to the study by Abdulkadiroglu et al [1] this is the only way to keep the mechanism strategy-proof. In the Scottish Foundation Allocation Scheme [31], where the junior doctors are matched to hospitals, the organizers attempt to break the ties in such a way that in the resulted matching as many doctors are allocated as possible (see Irving and Manlove [17]).<sup>5</sup>

In contrast, in the Hungarian higher education admission scheme [30] the ties are not broken, therefore the students applying for a particular programme with equal scores are either all accepted or all rejected. We call this an *equal treatment* policy.

In particular, the ties are handled in the following way in Hungary. No quota may be violated, so the last group of students with the same score, with whom the quota would be exceeded, are all rejected. There is however an alternative policy that could be followed where the quotas may be exceeded by the admission of the last group of students with the same score, but only if there were unfilled places left otherwise.

As we will show in Section 4, both concepts can lead to matchings that satisfy special stability conditions based on score-limits that we formalize in Section 3. We refer to the first, more restrictive solution as *H-stable* (i.e., higher-stable) *score-limits* and we call the second, more permissive solution *L-stable* (i.e., lower-stable) *score-limits*. Note that these stable score-limit concepts generalize the original notion of stability by Gale and Shapley,

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<sup>4</sup>From the information published at the website of the Central Applications Office [28] it seems that the college-proposing Gale-Shapley algorithm is used in Ireland with some special features. One is that students can apply for 'level 8' and 'level 7/6' courses simultaneously, and these applications are processed separately, so a student may receive more than one offer at a time. There are deadlines for accepting offers and if offers are rejected then further offers are made by the higher education institutions, so the mechanism is somewhat decentralized. The tie-breaking is based on 'random-numbers' assigned to students with regard to each programme they applied for, so the ties are broken differently for different programmes involving perhaps the same applicants.

<sup>5</sup>In SFAS [31], applicants are ranked by NHS Education for Scotland in a so-called master list, in order of score each applicant has a numerical score allocated partly on the basis of academic performance and partly as a result of the assessment of their application form. The range of possible scores (approximately 40–100) is much smaller than the number of applicants (around 750 each year), so there are ties of substantial length in the master list.

since they are equivalent to that if no tie occurs. In Section 4, we show how one can extend the Gale-Shapley algorithm to find H-stable and L-stable score-limits. Moreover, in Section 5 we prove that the applicant-oriented versions provide the minimal stable score-limits (therefore they are the best possible solutions for the applicants), whilst the college-oriented versions provide maximal stable score-limits (therefore, they are the worst possible solutions for the applicants). We note that the above results are deducible from some general theorems on substitutable choice functions by Kelso and Crawford [18] and Roth [21], as it was very recently demonstrated by Fleiner and Jankó [13]. We describe these arguments in detail at the end of Section 5.

In Section 6 we show that comparing the H-stable and L-stable score-limits, the L-stable score-limits are more favorable for the applicants as they are lower. In particular, we show that no college can have a higher score-limit in the applicant-optimal L-stable solution than in the applicant-optimal H-stable solution (and the same applies for the applicant-pessimal solutions produced by the college-oriented versions). Interestingly, we also show that the applicant-optimal solution produced after a tie-breaking is always between these two kinds of solutions. Therefore the matchings corresponding to the H-stable and L-stable score-limits may provide upper and lower bounds for every applicant regarding her match in a scheme which uses any kind of tie-breaking strategy. Finally, in Section 7 we give examples showing that neither the H-stable nor the L-stable version of the applicant-oriented score-limit algorithm is strategy-proof. We conclude in Section 8.

## 2 Brief description of the application

In this section we briefly introduce the Hungarian higher education admission system. See more at the website of the European research network on Matching in Practice [32].

### Hungarian higher education admissions

In Hungary, higher education is free of charge in principle. There is, however, a quota for state financed places and all students who cannot fit in this quota (or want to do more than one study) have to pay some contribution. For indication on the numbers, in the last main matching round in 2011 the total number of applicants was 140954 and 125735 of them applied for state-financed places. The total number of students admitted was 98144 and 67035 of them got a state-financed place (thus around 31000 students were charged fees for their studies at programmes starting in September 2011).

Admissions have been organized via a centralized matching scheme since 1985. In the current system three matching rounds are conducted every year, starting from 2008. The main round is in spring, finishing with the announcement of the score-limits in July. There is an additional round at the end of the summer for unfilled programmes which start in September, and the third matching round is conducted in the winter for students who want to start their MSc studies in February. In 2011, the number of applicants in the above three matching rounds were 140954, 13294 and 6418, respectively.

The matching scheme is based on a centralized scoring system. The students apply for BSc or MSc programmes. Their scores are coming from their secondary school grades and from their maturity exams. Regarding the latter, students can choose between normal and high levels. Volunteering for high level exam may result in extra scores, but these are more difficult to pass. A new governmental regulation proposes to make high level exams compulsory in those subjects which are relevant for the programme the student applied for. So exams are centralized, but a student may have different scores for different programmes, as only the relevant subjects are considered (e.g. for computer science programmes the

grades and exam scores in physics are counted, but for economics history is considered instead, besides the main subjects - such as maths, Hungarian literature and grammar). Extra scores can be obtained if the applicant has a certificate in some languages, or had good results in national or international competitions (not just scientific, but also sports or art), or because of social and medical conditions (e.g. young mothers and disabled people get some priority). The final scores are integer numbers, currently limited to 480. Note that the maximum score was 144 until 2007 which resulted massive ties.

Note that the scoring system was less centralized before 2000. For example universities could even hold interviews. The reason for having a centralized scoring method based only on common exams is the presence of national quotas which are set by the government in each subject (e.g. in computer science only the best 3000 applicants can study for free every year). So the performance of these students, who want to get state-financed places in a subject, must be comparable even if they apply to different universities.

Students may apply for any number of programmes, although they are charged a fee for every item (about 10 EUR) after the third application. Actually, this might be a reason why the average length of the preference lists is 3.5 and not higher. The applicants should also indicate whether they are willing to pay the contribution or whether they are applying for a state-financed place regarding each programme in their lists.<sup>6</sup> In other words, students may be admitted to a programme under two kinds of contracts (either they pay a contribution or not) and their preference lists are on possible contracts. For simplicity, these contracts will be referred to as colleges interchangeably in the paper in order to keep the original terminology of Gale and Shapley.

The quotas are set by the universities in agreement with the responsible Ministry for each programme. A specificity of the Hungarian system is that universities can set also lower quotas for each programme they offer and if the lower quota is not filled then the programme is canceled. Besides the lower and upper quotas for each programme, which apply for both the state-financed and privately-financed students, there are upper quotas in each subject set by the government for the total number of students admitted for state-financed studies.

The centralized matching is run by a non-profit governmental organization and as a result they announce the score-limits for all programmes regarding both the state-financed and privately-financed places. Each student is admitted to the first programme on her list where she achieves the score-limit. Obviously, the score-limits for state-financed places are higher than for privately-financed places, so those who are willing to pay a contribution can get admitted more easily.<sup>7</sup>

The implemented algorithm was a generalized version of the college-oriented Gale-Shapley algorithm until 2007, and since 2007 the core of the matching procedure has been the applicant-oriented Gale-Shapley algorithm. There are at least four special features in this scheme that required an extension of the original algorithm with some extra heuristics.

1. Ties can occur, since students applying for the same programme may have equal scores. The attempted solutions are the so-called H-stable score-limits, which satisfy the condition that we cannot decrease the score-limit of any over-demanded programme without violating its quota. This means that the last group of student applying to a programme with the same score, with whom the quota would be

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<sup>6</sup>For example, the first choice of a student may be a state-financed place in an Economics BSc programme at university A, her second choice might be another state-financed place at university B but her third choice can be a privately-financed place in the Economics BSc programme again at university A and so on.

<sup>7</sup>An interesting by-product of the matching system is that the score-limits are actually very good indicators of the quality and popularity of the programmes, and they highly correlate with the students' preferences and also with the job market perspective of the graduates.

exceeded, are all rejected. This special feature is the subject of our paper.

2. In addition to the upper quotas some colleges may have lower quotas as well. This feature has been studied in [7]. The bad news is that for a reasonable stability concept the existence of a stable solution is not guaranteed any more, and the related problem is NP-hard.
3. Some sets of colleges may have common upper quotas. This feature has been studied also in [7]. The presence of common quotas does not necessarily ruin the nice properties of the college admissions problem. In fact, when the set system is so-called 'nested' then a stable solution is guaranteed to exist and a generalized Gale-Shapley method finds a stable solution. This was the case in the Hungarian application until 2007. Since then the corresponding set system is not nested any more, a stable solution may not exist and the related problem is NP-hard.
4. Students can apply for pairs of teaching programmes (e.g. to become teachers in both math and physics). This problem is closely related to a well-known problem of resident allocations with couples where junior doctors may form couples and submit joint applications for pair of positions. See a survey on the latter problem [8]. Note that this feature also implies that a stable solution may not exist and makes the related computational problem NP-hard.

Since the current model of the application embeds several NP-hard computational problems because of three special features from the above four, it is reasonable to use different heuristics in practice. The score-limit algorithm used by the central office is based on the applicant-proposing Gale-Shapley mechanism, which we will present in detail in the next sections.

### 3 The definition of stable score-limits

Let  $A = \{a_1, a_2, \dots, a_n\}$  be the set of applicants and  $C = \{c_1, c_2, \dots, c_m\}$  be the set of colleges, where  $q_u$  denotes the quota of college  $c_u$ . Let the ranking of the applicant  $a_i$  be given by a preference list  $P^i$ , where  $c_v >_i c_u$  denotes that  $c_v$  precedes  $c_u$  in the list, i.e. the applicant  $a_i$  prefers  $c_v$  to  $c_u$ . Let  $s_u^i$  be  $a_i$ 's final score at college  $c_u$ . Final scores are positive integers for all acceptable applicants, as in practice the students with scores below a common minimum threshold are rejected automatically (currently this minimum score is 240 in Hungary with a maximum score of 500, which applies for every study).

The *score-limits* of the colleges are represented with a non-negative integer mapping  $l : C \rightarrow \mathbb{N}$ . An applicant  $a_i$  is admitted to a college  $c_u$  if she achieves the score-limit at college  $c_u$ , and that is the first such place in her list, i.e. when  $s_u^i \geq l(c_u)$ , and  $s_v^i < l(c_v)$  for every college  $c_v$  such that  $c_v >_i c_u$ .

If the score-limits  $l$  imply that applicant  $a_i$  is allocated to college  $c_u$ , then we set the Boolean variable  $x_u^i(l) = 1$ , and 0 otherwise. Let  $x_u(l) = \sum_i x_u^i(l)$  be the number of applicants allocated to  $c_u$  under score-limits  $l$ .

Furthermore, let  $l^{u,t}$  be defined as follows:  $l^{u,t}(c_u) = l(c_u) + t$  and  $l^{u,t}(c_v) = l(c_v)$  for every  $v \neq u$ . That is, we increase the score-limit of college  $c_u$  by  $t$  (or decrease it if  $t$  is negative), but we leave the other score-limits unchanged.

To introduce the H-stable and L-stable score-limits, first we define the corresponding feasibility notions. Score-limits  $l$  are *H-feasible* if  $x_u(l) \leq q_u$  for every college  $c_u \in C$ . That is, the number of applicants may not exceed the quota at any college. This means that the last group of students with equal scores, with whom the quota would be exceeded, are

all rejected. Score-limits  $l$  are *L-feasible* if for every college  $c_u \in C$  such that  $x_u(l) \geq q_u$  it must be the case that  $x_u(l^{u,1}) < q_u$ . So the quotas may be exceeded at any college, but only with the worst group of students who are admitted there with equal scores.

We say that score-limits  $l$  are *H-stable* (resp. *L-stable*) if  $l$  are H-feasible (L-feasible) and for each college  $c_u$  either  $l(c_u) = 0$  or  $l^{u,-1}$  are not H-feasible (resp. L-feasible). Thus H-stability means that we cannot decrease the score-limit of any college without violating its quota assuming that the others do not change their limits. L-stability means that no college  $c_u$  can admit a student if at least  $q_u$  of its current assignees have a higher score, but otherwise the score limits must be as small as possible. H-stability is the concept that is currently applied in the Hungarian higher education matching scheme.

We note that if no tie occurs (i.e. every pair of applicants have different scores at each college), then the two feasibility and stability conditions are the same and they are both equivalent to the original stability concept defined by Gale and Shapley. The correspondence between stable score-limits and stable matchings in case of strict preferences was first observed by Balinski and Sönmez [5] in relation with the Turkish college admissions scheme (where ties do not occur due to a tie-breaking strategy based on the age of the applicants). Furthermore Azevedo and Leshno [2] have also used this observation in a general college admissions model involving continuum number of students.

## 4 Stable score-limit algorithms

Both the H-stable and L-stable score-limit algorithms are natural extensions of the Gale–Shapley algorithm. The only difference is that now, the colleges cannot necessarily select exactly as many best applicants as their quotas allow, since the applicants may have equal scores. If the scores of the applicants are all different at each college then these algorithms are equivalent to the original one. In this section we will present the applicant-proposing and the college-proposing score-limit algorithms. For simplicity we describe these algorithms with regard to the H-stability concepts only and we add some information about the L-stable versions in brackets whenever they differ from the H-stable versions.

### College-oriented algorithms:

In the first stage of the algorithm, let us set the score-limit at each college independently to be the smallest value such that, when all applicants are considered, the number of applicants offered places does not exceed its quota (resp. may exceed the quota but only if without the last tie of these students the quota is unfilled). Let us denote these score-limits by  $l_1$ . Obviously, there can be some applicants who are offered places by several colleges. These applicants keep their best offer, and reject all the less preferred ones, moreover they also cancel their less preferred applications.

In the subsequent stages, the colleges check whether their score-limits can be further decreased, since some of their offers may have been rejected in the previous stage, hence they look for new students to fill the empty places. So each college sets its score-limit independently to be the least possible that keeps the solution H-feasible (resp. L-feasible) considering their actual applications. If an applicant gets a proposal from some new, better college, then she accepts the best offer, at least temporarily, and rejects or cancels her other, less preferred applications.

Formally, let  $l_k$  be the score-limit after the  $k$ -th stage. In the subsequent stage, at each college  $c_u$ , the largest integer  $t_u$  is chosen, such that  $t_u \leq l_k(c_u)$  and  $x_u(l_k^{u,-t_u}) \leq q_u$  (resp. if  $x_u(l_k^{u,-t_u}) \geq q_u$  then  $x_u(l_k^{u,-t_u+1}) < q_u$ ). That is, by decreasing its score-limit by the largest score  $t_u$  that keeps the solution H-feasible (resp. L-feasible), i.e., where the

number of applicants offered a place by  $c_u$  does not exceed its quota (resp. may exceed the quota but only if without the last tie of these students the quota is unfilled), by supposing that all other score-limits remained the same. For each college  $c_u$  let  $l_{k+1}(c_u) := l_k^{u,-t_u}(c_u)$  be the new score-limit. Again, some applicants can be offered a place by more than one college, so  $x_u(l_{k+1}) \leq x_u(l_k^{u,-t_u})$ . Obviously, the new score-limits remain feasible.

Finally, if no college can decrease its score-limit then the algorithm stops. The H-stability (resp. L-stability) of the final score-limits is obvious by definition. Let us denote the corresponding solutions of the H-stable and L-stable versions by  $l_C^H$  and  $l_C^L$ , respectively.

### Applicant-oriented algorithms:

Let each applicant propose to her first choice in her list. If a college receives more applications than its quota, then let its score-limit be the smallest value such that the number of provisionally accepted applicants does not exceed its quota (resp. may exceed the quota but only if without the last tie of these students the quota is unfilled). We set the other score-limits to be 0.

Let the score-limits after the  $k$ -th stage be  $l_k$ . If an applicant has been rejected in the  $k$ -th stage, then let her apply to the subsequent college in her list, say  $c_u$ , where she achieves the actual score-limit  $l_k(c_u)$ , if there remains such a college in her list. Some colleges may receive new proposals, so if the number of provisionally accepted applicants exceeds the quota at a college (resp. exceeds the quota and without the last tie of these students the quota is still filled), then it sets a new, higher score-limit  $l_{k+1}(c_u)$ .

Again, for each such college  $c_u$ , this is the smallest score-limit such that the number of applicants offered a place by  $c_u$  does not exceed its quota (resp. may exceed the quota but only if without the last tie of these students the quota is unfilled), by supposing that all other score-limits remained the same. This means that  $c_u$  rejects all those applicants that do not achieve this new limit.

The algorithm stops if there is no new application. The final score-limits are obviously H-feasible (resp. L-feasible). The solution is also H-stable (resp. L-stable), because after a score-limit has increased for the last time at a college, the rejected applicants get less preferred offers during the algorithm. So if the score-limit in the final solution were decreased by one for this college, then these applicants would accept the offer, and the solution would not remain H-feasible (resp. L-feasible). Let us denote the corresponding solutions by the H-stable and L-stable applicant-oriented versions by  $l_A^H$  and  $l_A^L$ , respectively. The following result is therefore immediate.

**Theorem 4.1.** *The score-limits  $l_C^H$  and  $l_C^L$  obtained by the college-oriented score-limit algorithms are H-stable and L-stable, respectively. The score-limits  $l_A^H$  and  $l_A^L$  obtained by the applicant-oriented score-limit algorithms are H-stable and L-stable, respectively.*

## 5 Optimality of the outputs

It is easy to give an example to show that not only some applicants can be admitted by preferred places in  $l_A^H$  as compared to  $l_C^H$ , but the number of admitted applicants can also be larger in  $l_A^H$  (and the same applies for the L-stable setting). We say that score-limits  $l$  are *better* than  $l_*$  for the applicants if  $l \leq l_*$ , i.e., if  $l(c_u) \leq l_*(c_u)$  for every college  $c_u$ . In this case every applicant is admitted to the same or to a preferred college under score-limits  $l$  than under  $l_*$ .

**Theorem 5.1.** *Given a college admission problem with scores,  $l_C^H$  are the worst possible and  $l_A^H$  are the best possible stable score-limits for the applicants, i.e. for any H-stable score-limits  $l$ ,  $l_A^H \leq l \leq l_C^H$  holds.*

*Proof.* Suppose first for a contradiction that there exists a H-stable score-limit  $l_*$  and a college  $c_u$  such that  $l_*(c_u) > l_C^H(c_u)$ . During the college-oriented algorithm there must be two consecutive stages with score-limits  $l_k$  and  $l_{k+1}$ , such that  $l_* \leq l_k$  and  $l_*(c_u) > l_{k+1}(c_u)$  for some college  $c_u$ .

Obviously,  $l_k^{u,-t_u}(c_u) = l_{k+1}(c_u)$  by definition. Also,  $x_u(l_k^{u,-t_u}) \leq q_u < x_u(l_*^{u,-1})$ , where the first inequality holds by definition of  $t_u$ , as we choose the new limit for college  $c_u$  such a way that the number of temporarily admitted applicants does not exceed its quota. The second inequality holds by the H-stability of  $l_*$ . So there must be an applicant, say  $a_1$ , who is admitted to  $c_u$  at  $l_*^{u,-1}$  but not admitted to  $c_u$  at  $l_k^{u,-t_u}$ .

On the other hand, the indirect assumption implies that  $l_k^{u,-t_u}(c_u) = l_{k+1}(c_u) \leq l_*(c_u) - 1 = l_*^{u,-1}(c_u)$ . Applicant  $a_1$  has a score of at least  $l_k^{u,-t_u}(c_u)$ , which is enough to be accepted to  $c_u$ , so she must be admitted to some college  $c_v$  under  $l_k^{u,-t_u}(c_u)$  which is preferred to  $c_u$ . Obviously  $a_1$  must be also admitted to  $c_v$  under  $l_k$ . But the H-stability of  $l_*$  implies that  $l_*(c_v) > l_k(c_v)$ , a contradiction.

To prove the other direction, we suppose for a contradiction that there exists H-stable score-limits  $l_*$  and a college  $c_u$  such that  $l_*(c_u) < l_A^H(c_u)$ . During the applicant-oriented algorithm there must be two consecutive stages with score-limits  $l_k$  and  $l_{k+1}$ , such that  $l_* \geq l_k$  and  $l_*(c_u) < l_{k+1}(c_u)$  for some college  $c_u$ . At this moment, the reason for the incrementation is that more than  $q_u$  students are applying for  $c_u$  with a score of at least  $l_*(c_u)$ . This implies that one of these students, say  $a_i$ , is not admitted to  $c_u$  under  $l_*$  (however she has a score of at least  $l_*(c_u)$  there). So, by the H-stability of  $l_*$ , she must be admitted to a preferred college, say  $c_v$  under  $l_*$ . Consequently,  $a_i$  must have been rejected by  $c_v$  in a previous stage of the algorithm, and that is possible only if  $l_*(c_v) < l_k(c_v)$ , a contradiction.  $\square$

**Theorem 5.2.** *Given a college admission problem with scores,  $l_C^L$  are the worst possible and  $l_A^L$  are the best possible L-stable score-limits for the applicants, i.e. for any L-stable score-limits  $l$ ,  $l_A^L \leq l \leq l_C^L$  holds.*

*Proof.* Suppose first for a contradiction that there exist stable score-limits  $l_*$  and a college  $c_u$  such that  $l_*(c_u) > l_C^L(c_u)$ . During the college-oriented algorithm there must be two consecutive stages with score-limits  $l_k$  and  $l_{k+1}$ , such that  $l_* \leq l_k$  and  $l_*(c_u) > l_{k+1}(c_u)$  for some college  $c_u$ .

This assumptions imply that  $x_u(l_k^{u,-t_u+1}) < q_u \leq x_u(l_*)$ . Here, the first inequality holds by the L-feasibility of  $l_{k+1}$ , and the second inequality by the L-stability of  $l_*$ . At the same time, by our assumption,  $l_*(c_u) > l_{k+1}(c_u)$ , so  $l_*(c_u) \geq l_{k+1}(c_u) + 1 = l_k^{u,-t_u+1}(c_u)$ .

From the two above statements it follows that there must be an applicant, say  $a_1$ , who has a score  $s_u(a_1) \geq l_*(c_u)$  and is admitted to  $c_u$  under  $l_*$ , but is not admitted to  $c_u$  under  $l_k^{u,-t_u+1}$ . So  $a_1$  must have a seat at some college  $c_v$  under  $l_k^{u,-t_u+1}$  such that  $c_v >_{a_1} c_u$ . Obviously,  $a_1$  is also admitted to  $c_v$  under  $l_k$ . But  $a_1$  is not admitted to  $c_v$  under  $l_*$ , therefore  $l_k(c_v) < l_*(c_v)$ , a contradiction.

To prove the other direction, we suppose for a contradiction that there exist stable score-limits  $l_*$  and a college  $c_u$  such that  $l_*(c_u) < l_A^L(c_u)$ . During the applicant-oriented algorithm there must be two consecutive stages with score-limits  $l_k$  and  $l_{k+1}$ , such that  $l_* \geq l_k$  and  $l_*(c_u) < l_{k+1}(c_u)$  for some college  $c_u$ .

At this moment, the reason for the incrementation is that more than  $q_u$  students are applying for  $c_u$  with score at least  $l_*$ , and  $c_u$  can choose a new score-limit  $l_{k+1}(c_u) = l_k^{u,-t_u}(c_u)$ , where  $t_u > l_*(c_u) - l_k(c_u)$ .

This implies that one of those students, who are admitted by  $c_u$  under  $l_{k+1}$ , say  $a_1$ , is not admitted to  $c_u$  under  $l_*$ . However she has a score higher than score-limit  $l_*(c_u)$  there. So, by the L-stability of  $l_*$ , she must be admitted to a preferred college, say  $c_v$ , under  $l_*$ .

Consequently, in the applicant-proposing procedure  $a_1$  must have been rejected by  $c_v$  at some previous stage, and that is possible only if  $l_*(c_v) < l_k(c_v)$ , a contradiction.  $\square$

## General arguments with choice functions

As we mentioned in the Introduction, our results presented in Sections 4 and 5 are deducible from some general theorems on substitutable choice functions by Kelso and Crawford [18] and Roth [21], as Fleiner and Jankó [13] pointed out. The selection of the colleges can be described by their choice functions. For a college  $u$  and a set of applicants  $X$ , let  $Ch_u(X)$  denote the set of selected applicants. A choice function  $Ch_u$  is *substitutable* (or *comonotone*) if  $X \subseteq Y$  implies  $(X \setminus Ch_u(X)) \subseteq (Y \setminus Ch_u(Y))$ , which means that the set of applicants rejected from a set  $Y$  must be also rejected from its subset  $X$ . This condition holds with respect to both the L-stable and H-stable score limits. Kelso and Crawford [18] showed that if the choice functions are substitutable on both sides of a many-to-one markets then there always exists a stable matching, moreover there is one stable matching that is optimal for the colleges. Roth [21] showed the existence of an applicant-optimal matching for this model (and also for the more general many-to-many case).

Furthermore, Fleiner and Jankó [13] gave new results on the structure of stable matchings that applies for L-stable and H-stable score limits as well. They noticed that the choice function of the colleges under L-stability satisfy the *path-independence* property, that is for any set of applicants  $X \subseteq Y$ ,  $Ch_u(Y) \subseteq X \subseteq Y$  implies  $Ch_u(X) = Ch_u(Y)$ . Therefore the theorem of Blair [9] implies that the set of stable matchings corresponding to L-stable score-limits forms a lattice. However, the path-independence property does not hold for the choice functions related to H-stable score-limits. Yet, the stable matchings corresponding to H-stable score-limits form a lattice, as Fleiner and Jankó proved with the use of new concept, called *four-stability*.

## 6 Comparison of the H-stable and L-stable versions

Intuitively it seems that the L-stable version of the algorithm is more *applicant-friendly* than the H-stable version. It turns out that we can prove the following result.

**Theorem 6.1.** *The score-limits obtained in the L-stable version of the applicants-oriented procedure are always equal or lower than the score-limits obtained in the H-stable version of the applicant-oriented procedure: i.e.  $l_A^L \leq l_A^H$ .*

*Proof.* Part I. Some colleges may have number of admitted students less than or equal to their quota under  $l_A^H$ , i.e.  $q_u - x_u(l_A^H) \geq 0$ . Each college  $c_u$  has a "waiting" list of applicants, who would prefer to be admitted to  $c_u$  rather than to their currently assigned colleges.

Let us apply some random tie-breaking to the original preference relation of the colleges. Each applicant  $a_i$  will get a new score  $p_u^i \geq s_u^i$  such that no two applicants will have the same score at any college. Moreover, the new scores satisfy the following condition: if  $s_u^j < s_u^i$ , then  $p_u^j < p_u^i$ . These  $p_u^i$  scores are positive real numbers. For example, if there are three applicants with scores  $s_u^1 = s_u^2 = 1$ ,  $s_u^3 = 2$ , the new scores might be  $p_u^1 = 1$ ,  $p_u^2 = 1.5$ ,  $p_u^3 = 2$ .

After that the following procedure is organized. If the number of applicants on  $c_u$  college's waiting list is more than the number of empty seats then college  $c_u$  sets its new score-limit  $m_A^H(c_u) \leq l_A^H(c_u)$  equal to the score  $p_u^i$  of the last admitted applicant in its waiting list. Otherwise let  $m_A^H(c_u) = 0$ . Note that the new score-limits  $m_A^H$  are non-negative real numbers. This means that each college make offers to applicants from its waiting list who fit the new score-limit.

Some applicants may receive more than one proposal. Each applicant accepts one, from the most preferred college, and rejects the others. If there remain any empty seat in colleges then the second step is organized in the same manner and so on. Thus essentially we run a college-proposing deferred-acceptance procedure with regard to the new scores. At the end of this procedure some new score-limits  $m^R$  are achieved such that  $m^R \leq l_A^H$  by construction. These new score-limits  $m^R$  and the corresponding matching  $\mu^R$  are stable (in the Gale-Shapley sense) according to new strict preferences of colleges, also by construction.

Part II. For the strict preference profile and corresponding scores  $p_u^i$  from Part I we can organize applicant-proposing deferred acceptance procedure (which is, in case of strict preferences, equivalent to both the H-stable and L-stable applicant-oriented algorithms). The resulting matching  $\mu_A^R$  is, of course, stable under strict preferences. Furthermore, we can define score-limits  $m_A^R$  that are equal to the score of the last accepted applicant if college has no empty seats and to 0 otherwise. These score-limits  $m_A^R$  must be the lowest among all stable score-limits by the optimality theorem of Gale and Shapley. Therefore  $m_A^R \leq m^R$  in particular.

Part III. Now we deal with  $m_A^R$  score-limits. Let us get back to the original weak order preferences of the colleges and corresponding applicants' scores  $s_u^i$ . For each college with  $x_u(l_A^R) = q_u$  we can construct a "waiting" list of applicants, who prefer college  $c_u$  to their current matches under  $m_A^R$ .

Let us now apply the L-feasibility concept. At the first stage each college sets its new score-limit  $l_A^R(c_u) \leq m_A^R(c_u)$ , that is the largest value, which allows to admit equal or more than the quota under weak order preferences as L-feasibility prescribes. For example, if there are two applicants with the same score  $s_u^i$ , such that one of them is admitted to  $c_u$  under  $m_A^R$  and the other is on the waiting list then we have to 'treat them equally', so we should lower the score-limit. Each college makes offers to these additional applicants.

Some applicants may receive more than one offer from colleges; in this case each applicant chooses the most preferred college. After that if there is any college with number of admitted applicants less than its quota then a new round starts. Each college chooses new, lower, L-feasible limit, and so on. That is we run the college-proposing score-limit procedure under L-stability. At the end, some new score-limits  $l^L$  are achieved such that  $l^L \leq m_A^R$  by construction. These new score limits are L-feasible and L-stable, obviously.

Part IV. For each L-stable score-limit  $l^L$  we know that  $l_A^L \leq l^L$  from Theorem 5.2, where  $l_A^L$  are stable score-limits obtained by the L-stable applicant-oriented algorithm.

Now we can construct the following inequalities:  $l_A^L \leq l^L \leq m_A^R \leq m^R \leq l_A^H$ . So we can conclude that for any college admissions problem with score-limits the outcome by the L-stable applicant-oriented algorithm is better for the applicants (i.e. yields lower score-limits) than the outcome of the H-stable applicant-oriented algorithm. □

**Theorem 6.2.** *The score-limits obtained in the L-stable version of the college-oriented procedure are always equal or lower than the score-limits obtained in the H-stable version of the college-oriented procedure: i.e.  $l_C^L \leq l_C^H$ .*

*Proof.* Part I. Let us consider the  $l_C^L$  score-limits. Some colleges may have number of admitted students more than or equal to their quota,  $x_u(l_C^L) \geq q_u$ .

Let us apply a random tie-breaking to the original preference relation of the colleges. Each applicant  $a_i$  gets a new score  $p_u^i \geq s_u^i$  such that no two applicants have the same score at any college, and these new scores do not contradict with the original ordering. Moreover, if  $s_u^j < s_u^i$ , then  $p_u^j < p_u^i$ . These  $p_u^i$  scores are positive real numbers.

After that the following procedure is organized. At the first stage each college sets its new score-limit  $m_C^L(c_u) \geq l_C^L(c_u)$  such that according to the new scores  $p_u^i$  the number of applicants who fit this score-limit would be exactly  $q_u$ . The new score-limits  $m_C^L$  are non-negative real numbers. Let  $m_C^L(c_u)$  be equal to 0 if the number of students admitted to  $c_u$  is less than  $c_u$ 's quota and otherwise let  $m_C^L(c_u)$  be equal to the lowest score  $p_u^i$  of any admitted student.

Some applicants are rejected from colleges they were assigned under  $l_C^L$ . Each rejected applicant then applies to the subsequent college in her list. Colleges receive new applications and, if necessary, raise the limits so that number of accepted applicants are equal to their quota. Some new applicants may be rejected, so a second round is organized in the same manner and so on. Thus we run an applicant-proposing deferred-acceptance procedure with respect to the perturbed strict preferences. At the end, some new score-limits  $m^R$  are obtained such that  $m^R \geq l_C^L$  by construction. These new score-limits are stable (in the Gale-Shapley sense) according to the new strict preferences of colleges by construction.

Part II. For strict preference profile and corresponding scores  $p_u^i$  from Part I we can organize a college-oriented deferred-acceptance procedure. The resulting score-limits  $m_C^R$  are, of course, stable according to these strict preferences. Furthermore, the corresponding score-limits must be the lowest among all stable score-limits [14]. So,  $m_C^R \geq m^R$ .

Part III. Now we deal with  $m_C^R$  score-limits. For each college  $c_u$ ,  $x_u(l_A^R) \leq q_u$  holds under  $m_C^R$ . Each college  $c_u$  with number of assigned students lower than its quota has score-limit  $l_A^R(c_u) = 0$ . Now we get back to the original weak order preferences of the colleges and original applicants' scores  $s_u^i$ .

Let us now apply the H-feasibility concept. For each college we can construct a list of applicants, who prefer college  $c_u$  to their current matches under  $m_C^R$ . After that the following deferred acceptance procedure is organized. At the first stage each college sets its new score-limit  $l_C^R(c_u) \geq m_C^R(c_u)$  that is the smallest value, which allows to admit equal or less than the quota under weak order preferences as H-feasibility prescribes. Therefore some colleges may reject applicants. Each rejected applicant applies to the next college in her list. Colleges receive new applications and, if necessary, raise their score-limits in such a way that the number of accepted applicants is less than or equal to their quota. Some applicants may be rejected and a second round is organized in the same manner and so on. Thus we run an applicant-proposing deferred-acceptance procedure with regard to H-stability. At the end, each applicant is either accepted to some college or rejected by all acceptable colleges. Some new score-limits  $l^H$  are achieved such that  $l^H \geq m_C^R$  by construction. These new score-limits are H-feasible and H-stable, obviously.

Part IV. For each H-stable score-limit  $l^H$  we know that  $l_C^H \geq l^H$  from theorem 5.1, where  $l_C^H$  is a H-stable score-limit obtained by the applicant-oriented score-limit algorithm.

Now we can construct the following inequalities:  $l_C^L \leq m^R \leq m_C^R \leq l^H \leq l_C^H$ . So we can conclude that for any college admissions problem with score-limits the outcome by the L-stable college-oriented algorithm is better for the applicants (i.e. yields lower score-limits) than the outcome of the H-stable college-oriented algorithm. □

**Corollary 6.1.** *Applicant-optimal H-stable and L-stable score-limits ( $l_A^H$  and  $l_A^L$ ) are upper and lower bounds, respectively, for score-limits under any applicant-optimal stable matching with random tie-breaking. Similar statement applies for the applicant-pessimal score-limits ( $l_C^H$  and  $l_C^L$ ).*

Finally, we note that the proofs of Theorem 6 and 6.2 can be naturally extended to get an implication that is slightly more general than the above Corollary. Suppose that

the applicants could be strictly ranked by the colleges with exact scores  $e$ , but the scoring method used in the scheme is not *fine enough*, which leads to rounded scores  $s$  with ties. We say that scores  $f$  are *refinements* of scores  $s$  if  $f$  and  $s$  are not the same, and for any pair of applicants  $a_i$  and  $a_j$  and college  $c_u$ ,  $f_u^i = f_u^j$  implies  $s_u^i = s_u^j$  and  $s_u^i < s_u^j$  implies  $f_u^i < f_u^j$ . Intuitively, if there exist exact scores  $e$  then a scoring method is finer than another one if it produce scores closer to  $e$ .<sup>8</sup>

**Corollary 6.2.** *Suppose that the applicants can be ranked with exact scores  $e$  by the colleges, and the corresponding applicant-optimal stable score-limits are  $l_A(e)$ . Let  $s$  and  $f$  be scores such that  $e$  are refinements of  $f$  and  $f$  are refinements of  $s$ . Let  $l_A^H(s)$  and  $l_A^L(s)$  denote the applicant-optimal stable score-limits for scores  $s$  with respect to the  $H$ -stable and  $L$ -stable concepts. Then  $l_A^L(s) \leq l_A^L(f) \leq l_A(e) \leq l_A^H(f) \leq l_A^H(s)$  holds.*

## 7 Strategic issues

Here we give two examples showing that neither of the above described score-limit algorithms is strategy-proof. The manipulability from the applicants' side is only interesting in the case of applicant-oriented algorithms, as the applicants may successfully manipulate the college-oriented versions even for strict preferences (i.e., for scores with no ties). Therefore we only consider the applicant-oriented versions in the examples below.

**Example 1.** *Suppose that we have two colleges,  $c_u$  and  $c_v$  with one seat in each of them, and two applicants  $s_1$  and  $s_2$  applying to both  $c_u$  and  $c_v$  with a preference towards  $c_u$  and with equal scores at both places. So the preference list of the colleges and students are as follows.*

$$\begin{array}{ll} a_1 : c_u, c_v & c_u : (a_1, a_2) \\ a_2 : c_u, c_v & c_v : (a_1, a_2) \end{array}$$

Figure 1: An example for the manipulability of the  $H$ -stable applicant-proposing algorithm

*Here the only stable solution is the empty matching (i.e., score-limits higher than the scores of  $a_1$  and  $a_2$  at both colleges). However, if either of the students, say  $a_1$  withdraws her application at  $c_u$  then the unique  $H$ -stable solution (under falsified preferences) is matching where  $a_1$  is allocated to  $c_v$  and  $a_2$  is allocated to  $c_u$ . So the manipulator (and actually the other student also) would improve.*

The following example is essentially the same as the one that Hatfield and Milgrom [15] constructed in a different setting but for a similar purpose.

**Example 2.** *Suppose that we have two colleges,  $c_u$  and  $c_v$  with one seat in each of them, and three applicants  $a_1$ ,  $a_2$  and  $a_3$  applying to both  $c_u$  and  $c_v$  with the following scores,  $s_u^1 = 1$ ,  $s_u^2 = 1$ ,  $s_u^3 = 2$ ,  $s_v^1 = 3$ ,  $s_v^2 = 2$  and  $s_v^3 = 1$ . These can be described equivalently with the preference lists below.*

*Here the only  $L$ -stable solution is the matching  $\{(a_1, c_v), (a_3, c_u)\}$  (i.e., with score-limits  $l(c_u) = 2$  and  $l(c_v) = 3$ ). However, if  $a_2$  were to reverse her preferences with regard to the two colleges then the  $L$ -stable applicant-oriented algorithm would produce the matching  $\{(a_1, c_u), (a_2, c_u), (a_3, c_v)\}$ , where the manipulator (and actually both the two other applicants) would improve.*

<sup>8</sup>In Hungary the scoring method became finer in 2007. Until 2007 each written exam with a maximum score of 100 had been rounded to an integer score between 0 and 15. Since 2007 the exact score of these written exams are considered when calculating the final scores of the students. As a result the maximum score increased from 144 to 480 and the ties became much smaller in the rankings of the colleges.

$$\begin{array}{ll}
a_1 : c_u, c_v & c_u : a_3, (a_1, a_2) \\
a_2 : c_v, c_u & c_v : a_1, a_2, a_3 \\
a_3 : c_v, c_u &
\end{array}$$

Figure 2: An example for the manipulability of the L-stable applicant-proposing algorithm

## 8 Further notes

Regarding the Hungarian application, we would like to conduct an experiment with real data and compute the four possible extreme stable score-limits, namely the applicant-optimal and applicant-pessimal score-limits under H-stability and L-stability. It would be also interesting to see how these concepts can be used in other settings, e.g. what could be the corresponding solutions for the Boston and New York high school matching programs.

Regarding the theoretical problems, we would like to investigate whether there is any structure behind the H-stable and L-stable score-limits. It would be also worth to study further the relation of solutions satisfying equal treatment policy and those produced by tie-breaking strategies.

Finally, some manipulable stable matching mechanisms turned out to be incentive compatible in large markets [23, 16, 19]. Our question is whether either the applicant-oriented stable score-limit algorithms is approximately strategyproof in large markets.

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