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MŰHELYTANULMÁNYOK **DISCUSSION PAPERS**

MT-**DP – 2011/23**

Notes on the Bankruptcy Problem: an Application of Hydraulic Rationing

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BUDAPEST, 2011

Discussion papers
MT-DP – 2011/23

Institute of Economics, Hungarian Academy of Sciences

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May 2011

ISBN 978-615-5024-59-7
ISSN 1785 377X

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Abstract

We offer a new approach to the well-known bankruptcy problem based on Kaminski's idea. With the help of hydraulic rationing we give a proof to Aumann and Maschlers theorem i.e. the consistent solution of a bankruptcy problem is the nucleolus of the corresponding game. We use a system of vessels and water and the principles of mechanics to show this fact. The proof is not just simple and demonstrative but also provides an insight how the nucleolus is constructed in such games.

Keywords: bankruptcy problem, nucleolus, hydraulic rationing

JEL Classification: C71

Acknowledgement:

Research for the paper was supported by project No. K 69027 of the Hungarian Scientific Research Fund (OTKA).

The author thanks the funding of the Hungarian Academy of Sciences under its Momentum Programme (LD-004/2010).

Jegyzetek a csődproblémáról: a hidraulikus számítógép egy alkalmazása

Fleiner Tamás - Sziklai Balázs

Összefoglaló

A tanulmány a csődprobléma egy új megközelítését tárgyalja Kaminski ötletének felhasználásával. A hidraulikus számítógép segítségével új bizonyítást adunk Aumann és Maschler híres tételére, miszerint a csődprobléma konzisztens megoldása a csődjáték nukleolusza. A mechanika alapelveit, egy edényrendszert és vizet felhasználva szemléltetjük ezt. A bizonyítás nem csak egyszerű és könnyen átlátható de betekintést nyújt abba is, hogyan épül fel a nukleolusz hasonló játékokban.

Tárgyszavak: csődprobléma, nukleolusz, hidraulikus számítógép

JEL kód: C71

Notes on the Bankruptcy Problem: an Application of Hydraulic Rationing

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Abstract

We offer a new approach to the well-known bankruptcy problem based on Kaminski's idea. With the help of hydraulic rationing we give a proof to Aumann and Maschlers theorem i.e. the consistent solution of a bankruptcy problem is the nucleolus of the corresponding game. We use a system of vessels and water and the principles of mechanics to show this fact. The proof is not just simple and demonstrative but also provides an insight how the nucleolus is constructed in such games.

1 Introduction

The bankruptcy problem is one of the oldest in the history of economics. In the simplest form we have a firm which goes bankrupt and there are creditors who wish to collect their claims. The amount they demand exceeds the firm's liquidation value. The natural question arises: how to divide this value among the claimants? Depending on our notion of fairness we can impose many rules for such a division. For an an excellent survey on this matter see Thomson [11].

One of the oldest concepts can be found in the Talmud [10]. The proposed solution is puzzling at first nevertheless it exhibits many nice properties. It is a mixture of the concepts of constrained equal losses and constrained equal awards. The underlying rationing

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was not clear until Aumann and Maschler solved the riddle in 1985 [2]. They gave an elementary proof showing that each bankruptcy problem has a so called CG-consistent solution. Moreover these coincide with the examples in the Talmud. They also showed that the CG-consistent solution is the nucleolus of the corresponding bankruptcy game. This is an interesting result that comes with a less elementary proof.

In 2000 Kaminski introduced a fascinating new concept to represent the bankruptcy game and other similar problems: the hydraulic rationing [8]. A physical device consisting of vessels and water correspond to the claimants and the firm's liquidation value.

Here we show an elementary proof of the nucleolus property using Kaminski's idea. Our proof seems to be more direct than that of Benoit published in 1997 [3].

2 The Model

Here we describe the basic notions of the bankruptcy problem. Readers who are familiar with these concepts might want to skip this part and jump to the next section. For more on the models of fair allocation see [12].

Let $N = \{1, 2, \ldots, n\}$ be a set of agents. The **bankruptcy problem** is defined as a pair (c, E) where $E \in \mathbb{R}_+$ is the firm's liquidation value (or *estate*) and $c \in \mathbb{R}^n_+$ is the collection of claims with $\sum_i^n c_i > E$. Let $\mathbb{B}$ denote the class of such problems. A solution of a bankruptcy problem is a vector $\mathbf{x} \in \mathbb{R}^n_+$ with $x_i \leq c_i$ for any $i \in N$ which satisfies the condition that $\sum_i^n x_i = E$. For convenience' sake we introduce the notations $x(S) = \sum_{i \in S} x_i$ and $c(S) = \sum_{i \in S} c_i$ for any $S \subseteq N$. Hence the above conditions can be written as $x(i) \leq c(i)$ for any $i \in N$ and $x(N) = E$. A **rule** $r : \mathbb{B} \to \mathbb{R}^n$ is a mapping that assigns a unique solution to each bankruptcy problem. A dual of a rule r is denoted by r^*. The dual assigns awards in the same way as r assigns losses namely $r^*(c, E) = c - r(c, c(N) - E)$. A self-dual rule is one with $r^* = r$, such rule treats losses and awards in the same way.

Now we formalize several rules which we will need later on. The reader can find several characterizations of these rules in [6] and [9].

The **constrained equal-awards** (CEA) rule assigns equal awards to each agent subject to no one receiving more than his claim. The dual of this rule is the **constrained equal-losses** (CEL) rule. In this case losses are distributed as equally as possible subject to no one receives a negative amount. Formally:

Constrained equal-awards: For all $(c, E) \in \mathbb{B}$ and $i \in N$, $CEA_i(c, E) = min(c_i, \lambda)$ where λ solves $\sum min(c_i, \lambda) = E$.

Constrained equal-losses: For all $(c, E) \in \mathbb{B}$ and $i \in N$, $CEL_i(c, E) = min(0, c_i - \lambda)$ where λ solves $\sum min(0, c_i - \lambda) = E$.

Another well-known rule is the **random arrival rule**. Suppose the claims arrive sequentially and they are fully compensated until the money runs out. The random arrival rule computes awards vectors for every possible ordering of claims and then takes the average. Hence it produces the same awards vector as the Shapley-value applied to the corresponding bankruptcy game[10].

The **Contested Garment Principle** is a division formula which can be derived from the Talmud. There are two claimants and a divisible good. According the principle each claimant should give the part of the good that he does not contest to the other claimant. Then the rest is split up equally.

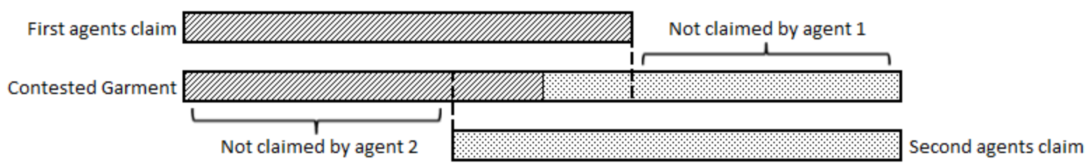

Figure 1: The Contested Garment Principle

Definition 1. *Let (c, E) be a bankruptcy problem. A solution is called CG-consistent or simply consistent, if for all $i \neq j$ the division of $x(i) + x(j)$ prescribed by the contested garment principle for claims c_i, c_j is $(x(i), x(j))$.*

The CG-consistent rule is the one that assign the CG-consistent solution to each bankruptcy problem. Sometimes it is also referred as **Talmud rule**. Formally it can be written as

$$T_i(c, E) = \begin{cases} \min\{c_i/2, \lambda\} & if\ E \leq \frac{1}{2}c(N), \\ \max\{c_i/2, c_i - \mu\} & if\ E > \frac{1}{2}c(N), \end{cases}$$

where λ and μ are chosen so that $\sum_{i \in N} T_i(c, E) = E$.

Observe that the Talmud rule is the combination of the constrained equal awards and the constrained equal losses rules.

3 Hydraulic rationing

Proof techniques that use the principles of mechanics were very common in the ancient times[3]. The increasing number of examples in the literature shows that they are no less useful today. Just to mention some well-known instances: the shortest path in a

[3] Archimedes wrote to Eratosthenes: "I thought fit to write out for you and explain in detail... a certain method, by which it will be possible for you... to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves..."

directed graph can be found using a system of strings and knots, congestion games can be represented by electric circuits, and - as in our case - rationing problems can be modeled using hydraulic computers.

To prove Aumann and Maschlers theorem we will employ Kaminski's idea and construct a specific hydraulic. In this hydraulic every claim is represented by a vessel while the firms liquidation value corresponds to the amount of water we pour into this system. Our vessels have a peculiar hourglass-shape with the following characteristics:

- Each vessel has an upper and lower tank of a shape of a cylinder.
- The upper and lower tanks have the same volume and they are connected with a capillary.
- The capillaries have negligible volume.
- The cylinders have a circular base with area of 1.
- The volume of vessel i is equal to the size of agent i's claim.
- Finally each vessel has the same height denoted by h. We may assume $h = c_{max}$ where c_{max} denotes the largest claim.

Note that the last condition implies that the vessel with the largest volume has no capillary part. We say that a hydraulic is **talmudic** if it incorporates the above characteristics.

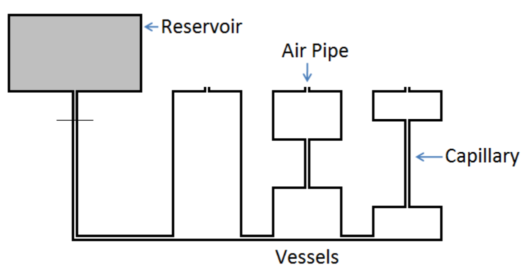

Figure 2: A connected talmudic hydraulic

It is not included in the above list but for a proper representation every vessel has to have also a capillary on the top tank where the air can leave the vessel. During the proof we will need two types of vessel systems. In a **connected hydraulic** vessels are connected with capillaries at the bottom. In this way if we pour water into any of the vessels each vessel starts to fill up. In a **disconnected hydraulic** different vessels can have different 'water levels' depending on how much water we pour into them. We did not define what do we understand under water level yet so let us do it here. The water level of a talmudic vessel i is denoted by I_i and defined the following way

$$I_i = \begin{cases} x(i) & if\ x(i) < c(i)/2, \\ [c(i)/2, h - c(i)/2] & if\ x(i) = c(i)/2, \\ x(i) + h - c(i) & if\ x(i) > c(i)/2. \end{cases}$$

Note that I_i is point valued whenever the ith agent receives less or more than one half of his claim. If he gets exactly half then the actual water level falls somewhere on the capillary part between the two tanks. Since the volume of the capillary is negligible we cannot tell the exact water level. Hence I_i is interval valued if $x(i) = c(i)/2$. In a connected hydraulic it is meaningful to speak about the common water level. Let z denote the common water level in a connected vessel system. Then

$$z \in \bigcap_{i=1}^{n} I_i.$$

Note that in a talmudic hydraulic z is always point valued since the vessel with the largest volume has no capillary part. We say that a hydraulic $\mathcal{H}$ **corresponds** to a bankruptcy problem (c,E) if the following conditions hold

- $\mathcal{H}$ has n vessels
- the volume of the ith vessel is equal to the size of the claim of agent i
- there is E amount of water distributed among the vessels.

A hydraulic that corresponds to a bankruptcy problem always implicitly defines an allocation rule. The nature of the rule depends on the shape of the vessels.

Here we show the representation of the rules we have already mentioned.

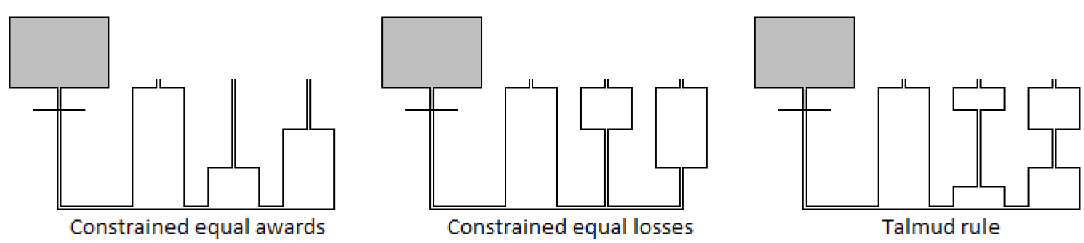

Figure 3: The representation of CEA, CEL and Talmud rules

Now we are ready to make some simple observations. For the next few statements we will not provide rigorous proofs only sketches.

Observation 1. ***(Kaminski)*** *Let (c, E) be a two-person bankruptcy problem and let $\mathcal{H}$ be the corresponding connected talmudic hydraulic. The solution x defined by the common water level z in $\mathcal{H}$ is consistent.*

Indeed it is not hard to check that the common water level z induces a consistent solution. For a detailed explanation the reader is referred to [1].

Corollary 1. *Each bankruptcy problem has a unique consistent solution.*

Consider the bankruptcy problem (c, E) and the corresponding connected talmudic hydraulic $\mathcal{H}$ with a common water level z. A CG-consistent solution means that no matter how we choose two vessels, the distribution of water between them is consistent. Since we have a connected hydraulic the water level is the same in every vessel. Furthermore z remains a common water level even if we disconnect a vessel from the system. Therefore the existence of a consistent solution follows from the two-vessel case. For the uniqueness part we proceed by contradiction. Let us assume that x is a consistent solution induced by z and $y \neq x$ is also consistent. Then there exists a vessel in y where the water level is below z and another vessel where the water level is above z. This is a contradiction. □

To further illustrate the robustness of Kaminski's method we show an interesting property of consistent solutions.

Observation 2. *A rule is self-consistent if and only if it corresponds to a connected hydraulic in which a shape of a vessel depends only on the respective claim size.*

Self-consistency means that a connected hydraulic will correspond to a rule even after we disconnect some of its vessels. Some rules like the Random arrival rule are not self-consistent. This means that a hydraulic which corresponds to the random arrival rule will not continue to do so after removing a vessel from the system. The shape of the remaining vessels has to change first to adapt to the new situation.

Another remarkable feature of the hydraulic approach is that it makes self-duality very apparent.

Observation 3. *The Talmud rule is a self-dual rule.*

Let $\mathcal{H}$ be a connected talmudic hydraulic with E amount of water in it. We have to show that the corresponding rule T is a self-dual rule formally

$$T^*(c, E) = c - T(c, c(N) - E).$$

We can translate this into the language of hydraulics. Let $x = (x_1, x_2, \ldots, x_n)$ be the solution induced by the common water level. Now consider a copy of $\mathcal{H}$ which is fully filled with water. Let $y = (y_1, y_2, \ldots, y_n)$ be the distribution of air when we let out E amount of water from the fully filled hydraulic. As the figure shows the distributions are the same.

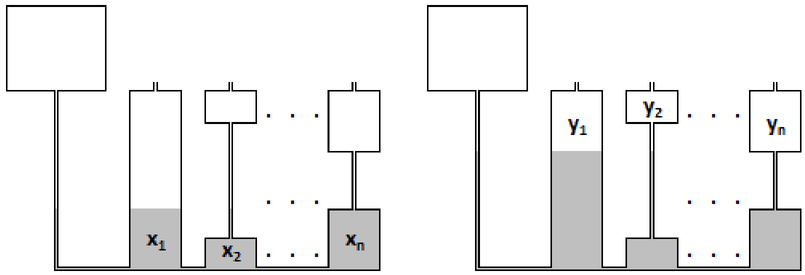

Figure 4: The hydraulic $\mathcal{H}$ with E and $c(N) - E$ amount of water in it.

Actually more is true. Easy to conclude the following fact.

Observation 4. *A rule is self-dual if and only if it corresponds to a horizontally symmetric connected hydraulic.*

4 The Nucleolus of the Bankruptcy Game

Bankruptcy problems can be modeled as coalitional games. Let us remind the reader to some basic notions of cooperative games. A cooperative game in characteristic function form is an ordered pair (N, v) consisting of the player set $N = \{1, 2, \ldots, n\}$ and a characteristic function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. Furthermore $x \in \mathbb{R}^n$ is called an allocation if it is efficient i.e. $x(N) = v(N)$. We say that an allocation x is an imputation or individually rational if $x(i) \geq v(i)$ for all $i \in N$. The set of imputation is denoted by $I(N, v)$.

$$I(N, v) = \{x \in \mathbb{R}^n \mid x(N) = v(N),\ x(i) \geq v(i) \ for\ all\ i \in N\}.$$

Given an allocation $x \in \mathbb{R}^n$, we define the excess of a coalition S as

$$e(S, x) := x(S) - v(S).$$

Let $\theta(x) \in \mathbb{R}^{2^n}$ be the excess vector that contain the 2^n excess values in a non-decreasing order.

We say that a vector $x \in \mathbb{R}^m$ is lexicographically less or equal than $y \in \mathbb{R}^m$, denoted by $x \preceq y$, if either $x = y$ or there exists a number $1 \leq j \leq m$ such that $x_i = y_i$ if $i \leq j$ and $x_{j+1} < y_{j+1}$.

Definition 2. *The nucleolus is the set of allocations of a game $x \in \mathbb{R}^n_+$ that lexicographically maximize $\theta(x)$ over (N, v). In other words*

$$N(v) = \{x \in I(N, v) \mid \theta(y) \preceq \theta(x) \ for\ all\ y \in I(N, v)\}$$

Now we are ready to introduce the notion of bankruptcy game. Let $N = \{1, 2, \ldots, n\}$ be a set of agents and $(c, E) \in \mathbb{B}$ a bankruptcy problem. For any $S \in 2^N$ the characteristic function of the related bankruptcy game is

$$v_{(c,E)}(S) = \max\left(E - c(N \setminus S), 0\right)$$

The characteristic function represent the worth of a coalition. By definition it is the value what is left of the firm's liquidation value $E = v(N)$ after the claim of each agent of the complement coalition $N \setminus S$ has been satisfied. This is the value the coalition can get without any effort. The excess of a coalition is defined by

$$e(x, S) = x(S) - v_{(c,E)}(S).$$

If a coalition S has nothing after all other claimants outside the coalition have been paid off then its excess will be $x(S)$. Otherwise the gain of S should be decreased by $v_{(c,E)}(S)$ since S would get $v_{(c,E)}(S)$ anyway.

We need two small observations.

Lemma 1. *Let $v_{(c,E)}$ be a bankruptcy game on player set N and x an imputation. The excess of $S \subseteq N$ can be written as*

$$e(x, S) = \min\left(x(S), c(N \setminus S) - x(N \setminus S)\right). \tag{1}$$

Proof of Lemma 1

By definition the excess of set S is

$$e(x, S) = x(S) - v_{(c,E)}(S) = x(S) - \max\left(E - c(N \setminus S), 0\right).$$

If $v_{(c,E)}(S) = 0$ then $c(N \setminus S) \geq E$. Subtracting $x(N \setminus S)$ from both sides yield

$$c(N \setminus S) - x(N \setminus S) \geq E - x(N \setminus S) = x(S) = \min\left(x(S), c(N \setminus S) - x(N \setminus S)\right).$$

On the other hand if $v_{(c,E)}(S) = E - c(N \setminus S)$ then $c(N \setminus S) \leq E$. Similarly we can gather

$$x(S) = E - x(N \setminus S) \geq c(N \setminus S) - x(N \setminus S) = \min\left(x(S), c(N \setminus S) - x(N \setminus S)\right).$$

□

We can translate (1) to the language of hydraulics. This time let $\mathcal{H}$ be a disconnected hydraulic. Now $x(S)$ is the amount of water that is distributed among the vessels that belong to coalition S. The excess of S is the minimum of the following two amount: the water contained in S or the air contained in $N \setminus S$.

Lemma 2. *Let* (c, E) *be a two-person bankruptcy problem. The closer an allocation is to the consistent solution the greater the excess vector is lexicographically. In particular the consistent solution of* (c, E) *is the nucleolus of the corresponding bankruptcy game.*

Proof of Lemma 2

Using the notions of cooperative games we can represent the contested garment principle in the following way.

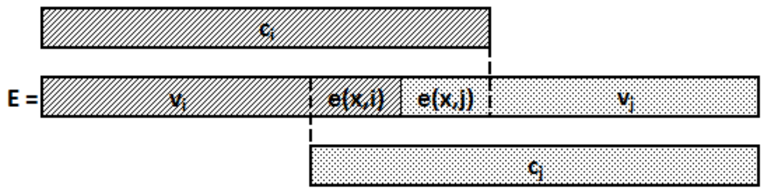

Figure 5: The doubly contested part is divided equally.

No matter how we divide the estate up we cannot give agent i less that v_i or more than c_i. Respectively we cannot give agent j less that v_j or more than c_j. Since v_i and v_j are fixed we can distribute only $e(x, i) + e(x, j)$ amount. Note that this sum of excess values is a constant. The gain of i is the loss of j and vice verse. By definition the consistent solution is the one where $e(x, i) = e(x, j)$. Using Observation 1 we can conclude that the common water level corresponds to a solution where the excesses are equal. Pouring water from vessel j to vessel i means we increase the excess of agent i and decrease the excess of agent j. $\square$

Now we are ready to prove the theorem of Aumann and Maschler.

Theorem 1. *The consistent solution of a bankruptcy problem is the nucleolus of the corresponding game.*

Proof of Theorem 1

Let there be given a bankruptcy problem (c, E) and a corresponding disconnected talmudic hydraulic $\mathcal{H}$. We proceed by contradiction. Let us assume that imputation x is the nucleolus of the game and it differs from the CG-consistent solution. Let z be the common water level that could have been obtained if the hydraulic was connected. We know that $\bigcap_{i=1}^{n} I_i = \emptyset$ otherwise x would coincide with the consistent solution. It follows that there exists $i, j \in N$ such that $I_i \cap I_j = \emptyset$ i.e. they have different water levels. Now consider this subsystem of vessels i and j. Without loss of generality we can assume that $I_i > z > I_j$ so the water level is higher in vessel i. We pour water from vessel i to vessel j until the water level in one of the vessels reaches z. Suppose we obtained allocations $x'(i)$ and $x'(j)$. Denote x' the imputation that coincides with x except in the ith and jth coordinate where it takes the values $x'(i)$ and $x'(j)$. We will show that $\theta(x) \prec \theta(x')$.

First observe that the excess of those coalitions that contain both i and j has not changed. Particularly let S be such a coalition. It is evident that S contains the same amount of water and $N \setminus S$ contains the same amount of air. The same goes for coalitions that contain neither i nor j. Let denote $S_{j,\bar{i}}$ the coalitions that contain j but does not contain i. After we poured some water from vessel i to vessel j the excesses are strictly greater for coalitions that belong to $S_{j,\bar{i}}$. Therefore it is enough to prove that

$$\min_{S \in S_{i,\bar{j}}} e(x', S) \geq \min_{S \in S_{j,\bar{i}}} e(x', S). \tag{2}$$

In this way we ensure that the excess vector has lexicographically increased. The excess is either the amount of water in S, namely $x(S)$ or the amount of air in the complement coalition $N \setminus S$, namely $c(N \setminus S) - x(N \setminus S)$. We are searching for the minimum of these two amounts. Observe that the smaller the coalition S is the smaller the value $x(S)$ is. Moreover the bigger the coalition S is the smaller the value $c(N \setminus S) - x(N \setminus S)$ is. Therefore looking at the coalitions of $S_{i,\bar{j}}$ either $\{i\}$ or $N \setminus \{j\}$ has the minimum excess. On the other hand looking at coalitions $S_{j,\bar{i}}$ either $\{j\}$, or $N \setminus \{i\}$ has the minimum excess. Therefore (2) immediately follows from

$$\min(x'(i), c_j - x'(j)) \geq \min(x'(j), c_i - x'(i)).$$

The left hand side of the inequality is nothing else than the excess of agent i after the water exchange. Similarly the right hand side is the excess of agent j. We already know from Lemma 2 that in such cases $e(x', \{i\}) \geq e(x', \{j\})$. Therefore we can conclude that $\theta(x) \prec \theta(x')$. This contradicts the fact that the imputation x is the nucleolus of the game. Therefore we have to reject our initial assumption that the nucleolus and the consistent solution differ.□

Remarks: Note that the proof works even without introducing hydraulic constructions. Some statements, like observation 1 and 3 are more transparent this way. However the real advantage of using hydraulics is that it makes easier to interpret notions like excess and nucleolus and gives many ideas what statements are true and how to prove them.

5 Algorithmic Aspects

The above proof also gives an algorithm how to reach the nucleolus from any initial imputation in at most $n - 1$ steps. In each step some of the lowest excesses increase while the loosing coalitions don't loose too much. It follows that there exists a linear time algorithm which calculates the nucleolus of any given bankruptcy problem. It was known before that there exists a polynomial time algorithm for computing the nucleolus of any convex cooperative game [7]. For more on the subject see [4] and [5]. Still a little bit surprising that the following simple pseudoalgorithm works:

INPUT: a set of agents $N = \{1, 2, \ldots, n\}$ a claims vector c and estate E.
OUTPUT: an imputation x that is the nucleolus of the corresponding game.
(1) If $E \leq c(N)/2$ distribute awards otherwise distribute losses.
(2) For $i = 1, 2, \ldots, n$ {
    If $|N| \cdot c_i/2 \leq E$ then $x_i := c_i/2$; $N := N \setminus \{i\}$; $E := E - x_i$}
(3) If N is non-empty then $x_i := E/|N|$ for $i \in N$.
(4) If losses were distributed then $x_i := c_i - x_i$ for $i = 1, 2, \ldots, n$.

Clearly each cycle in the for loop requires constant computation time. Phase 3 and 4 finish in at most n steps. Therefore the algorithm computes the nucleolus in $\mathcal{O}(n)$ time.

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