Power Networks – A network approach of voting theory

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Abstract

In this paper I study elections in electorates where preferences are defined by a social network. I argue that in some cases studying the ideology of the electorate is not the right way to understand the result of an election. In little electorates or in committees the personal connections can be much more important than political ideology. I develop a voting model where these personal connections add up to a social network and in this model I study the network properties and the social structures that lead to a voting equilibrium. I show that single peaked preferences on line and tree networks are inherent in my model, then I define a set of other networks where equilibrium can be guaranteed. I show that there are special network structures where the Condorcet winner cannot be identified by commonly used centrality measures (betweenness centrality and eigenvector centrality) but my model can identify it. Finally I calibrate my model to predict the victory of the Medici family in a medieval power struggle.

Key words: political economy, voting, social networks, citizen-candidate

JEL classification: D72, D85

1 Introduction

Voting theory focuses mostly on the relationship between the ideology of the electorate and the creed of the candidates. In this paper I argue that this approach might be successful in large electorates (e.g. national elections) but as the number of voters decreases, the importance of the personal ties increases. In little electorates where every voter knows personally the candidates the personal traits of the candidate are likely to be more important than the political ideology they represent.

The basic idea of my research is that society is not just a collection of individual voters: when we want to model social phenomena, studying the individual characteristics of the people in the society (e.g. general political preferences) gives us only the half of the picture, understanding the interpersonal relationships can be equally important. Members of a society are connected to each other with a great variety of different ties (e.g. family ties, friendship, coinciding interests, information flow) and these ties shape their preferences and (economic/political) decisions just as much as their individual characteristics do. The collection of these ties is the social network.

The social network is an illustrative tool to shed light to the differences between general political preferences and preferences based on personal ties that are typical in little electorates. In large electorates a political outcome, for example the electoral victory of candidate Anna, only affects voter Beth through the distance between the ideal policy of Beth and the policy actually implemented by Anna. However in little electorates the electoral victory of Anna can have direct effect on the voters regardless of the implemented policy[1]: with a non-infinitesimal probability candidate Anna and voter Beth are friends, relatives or tied with other interpersonal link. Being *close*[2] to the decision maker has its value in politics as it can lead to informal influence.

Not only the voter's informal influence over the decision maker can shape the elected decision maker's decisions, the decision maker's informal influence over some of her peers can be equally important. In constitutional democracies there is a set of decisions that the elected decision maker cannot make herself, she can only appoint people to make these decisions autonomously[3]. Trust, affection and the possibility of an informal influence surely plays a part in the appointment decisions of an elected decision maker. If Anna and Beth trust each other, Beth can have informal influence over Anna's decisions but in the same time she also can expect to be appointed to an autonomous decision making position as Anna expects to have an informal influence over the decisions of Beth. If we consider that being close to an appointed decision maker has similar perks (on a smaller scale) as being close to the elected one it is easy to see how political power spreads along social ties in the social network.

To study voting decisions and electoral outcomes in little electorates, I develop a

[1]To some extent this can be interpreted as a transferable ego-rent, see page 87 in Besley and Coate [1997]

[2]Closeness can be interpreted here as emotional closeness (affection, trust) or physical closeness: to spend time together or to have a word with each other.

[3]In less democratic regimes the delegation of decisions also appears (inevitably) because of the constraint of the decision-making capacity of the leader.

model of electoral competition where voting takes place in a society which is characterized by interconnected interests: the (electoral) success of a citizen yields profit, not only to herself, but also to those who are tied to her. The collection of these ties draws a social network and I use this network to study the result of a modified version of the three-stage voting game of the citizen-candidate literature: first the citizens choose whether or not to take the costly effort of running for office; then the citizens elect one representative from the self-selected pool of candidates; finally the winner receives a monetary prize (it represents the utility of the political power) that she shares with her neighbors in the network, who then share their profit with their own neighbors and so on (the political power spreads along the social ties).

My paper intends to link the citizen-candidate literature of voting theory[4] with the literature of social networks[5].

There are two empirical papers that are closest to my work, as both of them lies in the intersection of voting theory and social networks. Padgett and Ansell [1993] is a case study that shows how Cosimo de' Medici took advantage of the position of the Medici family in the marriage network of the Florentine nobility and managed to take over the Florentine city-state by being the single link towards the rest of the high society for a (sufficiently large) number of other noble families. Cruz et al. [2015] study the correlation between the candidate's position in the family network and its vote share (and winning probability) on the municipal election using data on family ties and the results of the 2010 municipal elections in the Philippines. They find that the more central candidates have higher vote share (and probability of winning the election). In this essay I build a theoretical model that can explain the empirical results of Padgett and Ansell [1993] and Cruz et al. [2015].

My first result shows that in any social network the voting game has a (trivial) pure strategy Nash equilibrium for some parameter values, where every citizen runs, they votes for themselves and win with equal probability; and that any social network with a strong Condorcet winner has a pure strategy Nash equilibrium, where the Condorcet winner runs uncontested, she gets all the votes and wins. Then I talk about two network properties that can guarantee the existence of equilibria with one or two candidates in some network classes and significantly simplify the problem in others. The first property is single-peakedness of the preferences and the second is balancedness of the network around "hub" players that act as bridges between sub-

[4]See Osborne and Slivinski [1996], Besley and Coate [1997] and Cadigan [2005].

[5]See Ballester et al. [2006], König et al. [2015], Galeotti and Mattozzi [2011], Elliott et al. [2014], Demange [1982], Demange [2011] and Buechel [2014].

groups in the network. I show that the Condorcet winner exists and that there are voting equilibria with one or two candidates in a wide set of networks: in the line (from Besley and Coate [1997]) and in the tree (from Demange [1982]) single peaked preferences are important, while in the bridge network (from Calvó-Armengol and Jackson [2004]) and in the windmill network (invented by me) the balancedness is crucial, in case of the social quilt networks (from Jackson et al. [2012]) both properties play a role. After that I compare my model's prediction with the prediction based on commonly used centrality measures and find examples of network structures where my model can identify the strong Condorcet winner but it is not the player with the highest centrality (betweenness or eigenvector centrality). I use these examples to point out that although betweenness and eigenvector centrality are highly correlated with the winner of the election in some special cases the analysis of the social structure has to go beyond centrality in order to provide a good prediction of the electoral outcome. Finally I use historical data to calibrate my model and predict the result of a power struggle in the medieval Florence: I find that in the social network of the Florentine nobility the Medici family was in a Condorcet winner position, so they had sufficient support to overthrow the ruling clique of the city.

The rest of the paper is organized in the following way: Section 2 introduces the model of the society and the voting game, Section 3 contains the main results of this essay, finally Section 4 concludes.

2 Model

In this section I present the main model of the paper[6]. In the first part I describe the way I model little electorates with a social network and how the voters' preferences over their peers' success are related to the social network. In the second part I describe the 3-stage citizen-candidate voting game: first the voters decide whether to present themselves on the ballot, then the voters vote on one of the self-selected candidates sincerely and finally the winner of the election assumes office and starts the chain of appointments (monetary transfers) as it is outlined in the introduction.

[6] In this model I built on the model of Elliott et al. [2014] that describes default cascades in financial networks.

2.1 Society

Let us consider a society of n citizens, where the set of citizens is denoted by $N = \{1, 2, 3, \dots n\}$. The society is structured by personal ties between pairs of citizens where these ties represent informal influences over each other's decisions consequently they proxy one's involvement in the other's success. The basic assumption of the model is that the citizens can not enjoy any kind of success alone: they necessarily share it partially with others. In this model success and political power (both formal and informal) are represented by monetary income that enters the society as a prize and spreads through "transfers".

Every citizen j keeps a positive share $\widehat{c_j}$ for herself of the income she receives either from outside of the society as a prize or from her peers as a transfer. Citizen j then distributes the rest of her income $(1 - \widehat{c_j})$ among the peers she is connected to[7], transferring a c_{ij} share to citizen i. Since citizen i receives a c_{ij} share of citizen j's income, to some extent, she becomes interested in j winning a prize.

The collection of the various c_{ij} links add up to the social network. In principle the social network consists of directed links ($c_{ij} \neq c_{ji}$). However, throughout the paper I assume that if there is a link with any positive weight from i to j then there is a positive link from j to i as well.

Assumption 1 *If there is a link from i to j with a positive weight then there is a link from j to i with positive weight: $c_{ij} > 0$ iff $c_{ji} > 0$.*

Assumption 1 means that the connections are reciprocal to some extent, but the fact that j has a big informal influence on i's decisions does not necessarily mean that i's opinion has a similarly large effect on j's decisions.

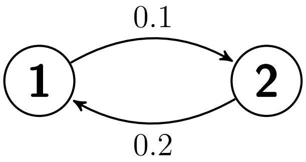
Matrix C is an $n \times n$ matrix that contains all the c_{ij} and so the characterizes the society by the transfer of money. It can be interpreted as a weighted adjacency matrix that defines the weighted graph C which represents the social network. The diagonal elements of matrix C are zero since there is no transfer from i to i. By definition every column j of C adds up to $1 - \widehat{c_j}$.

To understand better how the social structures relates to the matrix representations let us consider the simplest society with only two linked citizens in it ($N = \{1, 2\}$): Citizen 1 keeps 0.9, while Citizen 2 keeps 0.8 share of his income. They share the rest of their income [8]: Citizen 1 passes 0.1 share of his income to

[7]These transfers can be interpreted as the realization of a preexisting favor exchange agreement in an environment where it is technically impossible to directly enjoy all the perks of power.

[8]Possibly because they follow their preexisting favor exchange agreement.

Citizen 2, while Citizen 2 passes 0.2 share of his income to Citizen 1. The graph representation of the society is:



The corresponding C matrix (weighted adjacency matrix) and $\widehat{C}$ diagonal matrix[9] are:

$$C = \begin{pmatrix} 0 & 0.2 \\ 0.1 & 0 \end{pmatrix}, \quad \widehat{C} = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.8 \end{pmatrix}.$$

A direct consequence of Assumption 1 is that the appointments affects both the appointing and the appointed: when the elected decision maker j appoints citizen i to a decision making position she giver her political power (represented by c_{ij}) but also wins indirect influence over the position (represented by c_{ji}). Similarly i as appointed decision maker gains indirect influence by her appointees to lower level positions and this means a second degree indirect influence for j. In the model all these direct and indirect influences are represented by a series of money transfers: matrix C describes the transfer rule in each round. The initial income that comes outside of the society is distributed according to C (first degree indirect influence), then the beneficiaries of the first round share their income in the second round according to matrix C^2 (second degree indirect influence), matrix C^3 describes the third round (third degree indirect influence), and so on. As $\sum_{k=1}^{\infty} C^k$ converges to $(I - C)^{-1}$ the final distribution of political power (represented by monetary wealth) can be written as:

$$A = \widehat{C}(I - C)^{-1},$$

where $\widehat{C}$ is a diagonal matrix with the different $\widehat{c}$'s in its diagonal. In matrix A element a_{ij} is the final income that i receives (after accounting for all the back and forth transfers) if j wins a prize of value normalized to 1 from outside of the society. By construction matrix A is a column stochastic matrix (all its elements are between 0 and 1 and its columns add up to 1) and it is diagonally majored (in each row the diagonal element is the biggest). The C and the A characterizations are different

[9] $\widehat{C}$ is a diagonal matrix with the different $\widehat{c}$'s in its diagonal.

ways to describe the same underlying society and there is an unambiguous mapping between the two matrices.

Returning to the previous example with two citizens in the society, matrix A can be calculated as:

$$A = \widehat{C}(I - C)^{-1} = \begin{pmatrix} 0.918367 & 0.183673 \\ 0.0816327 & 0.816327 \end{pmatrix},$$

which means that if citizen 1 wins a prize of value 1 the wealth distribution would be 0.92 to citizen 1 and 0.08 to citizen 2, while if the winner is citizen 2 the distribution is 0.18 to citizen 1 and 0.82 to citizen 2. It worths to emphasize that the final outcome is different from what the citizens keep for themselves, $\widehat{C}$, and also from what they pass to their peers, C.

2.2 Voting game

After the careful definition of the social interactions let us consider a voting game in this society. The voting game is a competition for a prize of value 1 where payoff-maximizing actors play strategically and the winner is decided by the votes of the citizens of the society. The person who wins the election receives the prize that she shares with the peers she is connected to in the social network C, her peers do likewise with the transfers they receive from the winner, and so on, resulting in a final wealth distribution equal to the winner's corresponding column in matrix A. As this outcome vector is the monetary payoff that the society can expect in case of the victory of a given candidate, voters can derive their preferences over the competing candidates directly from matrix A.

On the election all citizens cast their vote sincerely for one of the candidates on the ballot. The model follows the citizen-candidate literature, so candidates are a self-selected group of citizens who decided to run for office and to pay a positive running fee β, which is strictly less than the prize ($0 < \beta < 1$). The group of candidates is represented by the binary vector $\mathbf{r}$, element $r_i = 1$ if citizen i decides to run for office and 0 otherwise. The election is decided by simple majority rule, in case of a tie in votes the winner is chosen with equal probability from the tying candidates. The expected outcome of the election is represented by the probability vector $\mathbf{e}$, element $e_i = 1$ if i wins the election alone, it is $e_i = 1/m$ if i is one of the m tying winners and 0 otherwise. Voting is sincere in the model which means that each voter automatically votes for the candidate that offers the most to her according to the corresponding row of A, and chooses with equal probability in case of a tie. For example citizen i prefers candidate l to k if $a_{il} > a_{ik}$, or generally speaking citizen i votes for candidate l in an election where l is in the candidate pool R and $a_{il} = \max_{j \in R}[a_{ij}]$. As a consequence of this, the only strategic decision of the players is the decision on running for office. The social structure, A, and the strategy profile, $\mathbf{r}$, together determine the (expected) political outcome, $\mathbf{e}$.

If no citizen decides to run for office the default outcome is implemented and its corresponding payoff vector is $\boldsymbol{\pi}_\mathbf{0}$ – throughout this paper I assume that the default option is undesired by everyone ($\boldsymbol{\pi}_\mathbf{0} < \boldsymbol{\pi}(\mathbf{r})$ for any $\mathbf{r} \neq \mathbf{0}$). Otherwise the payoff vector is given by:

$$\boldsymbol{\pi} = A\mathbf{e} - \beta\mathbf{r}.$$

The social structure is commonly known across the society. The timing of the game is the following: (1) citizens simultaneously make their running decision; (2) all the citizens vote for one of the candidates; (3) winner is announced, and the payoff realized. The solution of the model is a correspondence from the social structure represented by A to an equilibrium candidacy vector $\mathbf{r}$. The equilibrium concept in this essay is pure strategy Nash equilibrium, where the equilibrium is defined as a situation in which there is no candidate in $\mathbf{r}$ that could be better off in case of deviating alone by quitting the election and there is no non-candidate citizen that could be better off in case of deviating alone by running.

Proposition 1 *If the running fee β is sufficiently low the voting game always has an equilibrium where all the voters run and vote for themselves and win the election with equal ($1/n$) probability.*

Proof Matrix A is diagonally majored, which means that in every row of A the diagonal element a_{ii} is bigger than the rest of the elements ($a_{ii} > \max_{i \neq j}[a_{ij}]$). The winning premium of i is given by $prem_i = a_{ii} - \max_{i \neq j}[a_{ij}]$, and it is always positive. If the running fee is set such that $0 < \beta < \frac{\min_{l \in N}[prem_l]}{n}$, the costs of running are lower for every citizen than the winning premium in a situation where all the citizens win with equal probability, so no citizen has incentive to quit the election. As no citizen has incentives to quit, the strategy profile $\mathbf{r} = \mathbf{1}$ is a Nash equilibrium.□

Proposition 2 *Without loss of generality, if player 1 is a **strong Condorcet** winner in a network C, then $\mathbf{r} = (1, 0, \ldots, 0)$ and $\mathbf{e} = (1, 0, \ldots, 0)$ is a Nash equilibrium.*

Proof Player 1 has no incentive to quit the race since $\boldsymbol{\pi}_{\mathbf{0}} < \boldsymbol{\pi}(\mathbf{r})$, which means as every other citizen, player 1 is also worse off with the default outcome $\boldsymbol{\pi}_{\mathbf{0}}$. No other voter i has incentives to run as they can make no difference: since player 1 is a strong Condorcet winner she wins independently of the candidacy of i, and as i cannot make a difference in the outcome she will not be willing to pay the running fee either. □

3 Results

In the model every citizen i makes a strategic decision to run or not to run for an elected office in order to maximize her own payoff π_i, taking the running decisions of the others as given:

$$\max_{r_i \in \{0,1\}} \pi_i \Big|_{r_{-i}} = [A]_i \mathbf{e} - \beta r_i,$$

where $[A]_i$ is the ith row of matrix A. The payoffs that the society can expect is represented by the vector $\boldsymbol{\pi}$ and determined by $\boldsymbol{\pi} = A\mathbf{e} - \beta\mathbf{r}$. Since voters vote sincerely in the model the preference ordering of a voter i over the (potential) candidates is derived from $[A]_i$. This means that the running decision and the voting preferences are separable: if voter i prefers candidate j to k when $r_i = 0$ this preference ordering does not change when $r_i = 1$. The running decision affects the pool of electable candidates but does not affect the preference ordering of the voters over the potential candidates – the preference ordering is determined by A alone.

As we saw in the previous section matrix A can be written as $A = \widehat{C}(I - C)^{-1} = \widehat{C}T$. Since $\widehat{C}$ is a diagonal matrix (and we use it to multiply from the left side) $\widehat{C}$ multiplies every element of $[A]_i$ with the factor $\widehat{c_i}$, but it does not affect the relationship between the row elements in any of the rows. Consequently matrix $T = (I - C)^{-1}$ is the ultimate source of preference ordering in the model: if $t_{ij} > t_{ik}$ then $a_{ij} > a_{ik}$ and so in an election where j and k are running for office voter i would vote for candidate j. I call T the matrix of cardinal preferences, and the "bigger than", "smaller than" (">" and "<") relationships between the elements of the rows of T define the ordinal preferences of the voters over the candidates in the model.

The cardinal preferences of a voter i can be related to the cardinal preferences of her neighbors in the social network C, as it is written in Lemma 1.

Lemma 1 *The cardinal preference of a citizen i over (potential) candidate j is a linear combination of the cardinal preferences of i's neighbors over j, where the linear weights are the inlinks of i.*

Proof The matrix of cardinal preferences $T = (I - C)^{-1}$. I can multiply the equation by $(I-C)$ from the left, and the rearrange the equation to get $T = I+CT$. From this expression I can write i's cardinal preferences over j as $t_{ij} = 0+\sum_{k=1}^{n} c_{ik}t_{kj}$. Since c_{ik} is different from zero only for the neighbors of i, t_{ij} is the linear combination of the t_{jk} cardinal preferences of the neighbors of i over k and the linear weights are the c_{ik} inlinks of i. □

From Lemma 1 we know that the preferences of the individual voters can be expressed as a linear function of their neighbor's preferences. Furthermore this relationship can also be found between subgroups of the society (instead of individuals) and the people who are linked to the subgroup ("neighbors" of the group). Let G be a connected subgroup of the society N ($G \subset N$) such that $G^C = N \setminus G$ is also connected and $B_G \subset G^C$ is the set of border people who connect G to the rest of the society. Given G the set of border people is defined as:

Definition 1 *The* ***border players*** *of a group G are all the $i \in B_G$ such that $i \notin G$ and $c_{ig} > 0$ for some $g \in G$.*

By the definition of B_G, $c_{kl} = 0$ for every k and l pairs if $k \in G$ and $l \in M = N \setminus (G \cup B_G)$.

Lemma 2 *The cardinal preferences of any $i \in G$ over a candidate $j \in G^C$ can be written as the weighted sum of the cardinal preferences of the bordering nodes in B_G over j.*

Proof Without loss of generality let us suppose that $G = \{1, 2, 3, \ldots, g\}$ and $B_G = \{g+1, g+2, \ldots, g+b\}$ and $M = \{g+b+1, g+b+2, \ldots, n\}$, and G and $G^C = B_G \cup M$ are both connected subnetworks. As we saw in the proof of Lemma 1 $T = I+CT$, so the vector of cardinal preferences of the voters in subgroup G over a candidate j is $t_{Gj} = C_{GG}t_{Gj} + C_{GB}t_{Bj} + C_{GM}t_{Mj}$ where C_{GG} is the $g \times g$ upper left corner of matrix C that describes the connection structure of subgroup G, similarly C_{GB} is the $g \times b$ part of C that describes the inlinks of subgroup G from B_G and C_{GM} is the $g \times (n-b-g)$ lower left corner of C; t_{Bj} and t_{Mj} stands for the vector of cardinal preferences of people in B_G and the people in M (respectively) over j. Matrix C_{GM} is a zero matrix, since every $c_{kl} = 0$ where $k \in G$ and $l \in M$, the previous expression simplifies to $t_{Gj} = C_{GG}t_{Gj} + C_{GB}t_{Bj}$. This matrix equation can be solved for t_{Gj} and the result is $t_{Gj} = (I - C_{GG})^{-1}C_{GB}t_{Bj}$, where $(I - C_{GG})^{-1}$ is a $g \times g$ matrix. From this equation we have that the cardinal preferences in G can be written as the linear combination of the cardinal preferences in B_G. $\square$

The most important consequence of Lemma 2 is a special structure of inheritance of preferences in the network. Suppose there is a group G in the society, its corresponding set of bridge people B_G and two candidates j and k. If every bridge people in B_G prefers j to k (or $t_{ij} > t_{ik}$ for every $i \in B_G$) then all the voters in G will also prefer j to k ($t_{lj} > t_{lk}$ for every $l \in G$). This structure of inheritance of preferences is particularly important in networks where it is easy to define G groups with few (or even one) bridge people: if there is no candidate from group G all the voters in G will derive their voting decision from the preferences of the bridge people in B_G.

3.1 Single peaked preferences in line and tree networks

The line network is arguably the simplest networks class where all the nodes are only connected to two other nodes with positive link weight, forming a single line between two end nodes. Although it is hard to imagine an actual social network with such a strict architecture the idea of representing the society with a single line is well established in political economics. The tree network is a richer network class where the only restriction on the architecture is that the network cannot form loops, in other words every pair of nodes must be connected with exactly one path in the network.

A voter i has single peaked preferences over a set of potential candidates R ordered from 1 to r if her most preferred candidate is i and she will prefer candidate 2 to 1, and candidate 3 to 2 and candidate $j+1$ to j as long as $j < i$ and then her preferences change and prefers candidate k to $k+1$ for $k > i$. The preferences are single peaked in the society over the candidates in R if assuming a fix 1 to r ordering, every voter has single peaked preferences over the candidates in R.

Proposition 3 *Cardinal preferences are single peaked in a line network and along all the possible paths of any tree network.*

Proof Every citizen i prefers herself to any other citizen since T is a diagonally dominated matrix by construction, where the diagonal elements are the biggest elements in every row and so $t_{ii} > \max_{j \neq i}[t_{ij}]$. In the line network we can define the connected social group G_l (G_r) that includes i and everyone to the left (right) of i. Since group G_l (G_r) is connected to the rest of the society by r (l) who is i's neighbor to the right (left) the whole group inherits r's (l's) cardinal preferences over potential candidates from the rest of the society ($N \setminus G_l$ and $N \setminus G_r$, respectively). As voter r (l) prefers herself to anyone else in $N \setminus G_l$ ($N \setminus G_r$), i and the rest of group G_l (G_r) will do so as well. Repeating the argument with every possible voter we get that every i has their cardinal preference peak at their own position and on the left (right) half of the network i prefers the closer candidate to the further.

The same reasoning can be used in case of a tree network with the extension that every node i can have more than two (left and right) neighbors. □

The line network is an important network to study because it makes me able to connect my results to the traditional results of the voting theory (e.g. Besley and Coate [1997], Osborne and Slivinski [1996], Cadigan [2005]). Thanks to the single peaked preferences the median voter theorem can be used in the line network. According to the median voter theorem the median voter is a strong Condorcet winner if n is odd and the 2 median voters are weak Condorcet winners if n is even.

The class of tree networks is also interesting because it makes me able to connect my research to the literature of tree networks. As Theorem 1 of Demange [1982] shows any society where all the voters have single peaked preferences over all the (potential) candidates can be represented as a tree network. Theorem 2 of Demange [1982] guarantees that there is a Condorcet winner in the tree network with single peaked preferences. If the preferences are single peaked in a tree network there is a Condorcet winner in the network. There can be one strong Condorcet winner that beats every other potential candidate in a 2-candidate election or it can be the case that there are two weak Condorcet winners, a pair of neighbors, who beat everyone else in a 2-candidate election and tie against each other.

3.2 Bridge and windmill networks

Bridge network is a network class that was used in Calvó-Armengol and Jackson [2004] as an example of a society where the social groups are minimally connected.

Definition 2 ***Bridge network*** *is a social network that consists of two social groups where the groups are connected by only one link: this link is the* ***bridge link*** *and the two players it connects are the* ***bridge players****.*

Figure 1 shows an example of a bridge network where there are two social groups: group 1 consists of player 1-5 and group 2 consists of player 6-10. The two social

Figure 1: Example of a bridge network, link 1-6 is the bridge

groups are connected with each other only 1-6 link, which is the bridge link of this network.

Proposition 4 *In a bridge network where the two social groups are of equal size the bridge players are Condorcet winners.*

Proof Without loss of generality let us denote the two social group of equal size G_1 and G_2, and the bridge players of the groups b_1 and b_2 respectively. As the voters of G_2 are connected to G_1 through b_1 they inherit the preference ordering of b_1 over the potential candidates of G_1. This means that they prefer b_1 to any other candidate from G_1, since b_1 prefers herself to any other candidate. Consequently b_1 can beat any candidate from G_1 in a 2-candidate election: b_1 votes for herself and has the votes of the entire G_2 which is exactly 50%+1 vote. Repeating the argument with b_2 and G_1 we can see that the non-bridge players cannot be Condorcet winners. When the two bridge players b_1 and b_2 run against each other G_1 is connected to b_2 through b_1 (and *vice versa*) so the groups support their corresponding bridge player: as the two groups are of equal size this leads to a tie. Finally, no candidate from G_2 can beat b_2 in a 2-candidate election as all the voters in G_1 are supporting b_1 and with those 50% of the votes the worth possible result for b_1 is a tie. Repeating the argument with b_2 and G_1 we can conclude that the two bridge players are the only players in the social network that cannot be beaten by any other player in a 2-candidate contest. This means that the two bridge players b_1 and b_2 are Condorcet winners. □

Similarly to the class of bridge networks in the windmill networks there is also limited communication among the subgroups of the society. However in the windmill networks the communication is maintained by a middle man, or hub player, that does not belong to any of the groups but has links to members in various groups.

Definition 3 ***Windmill network*** *is a social network that consists of several social groups where the social groups are only connected through a* ***hub player*** *that does not belong to any of the social groups.*

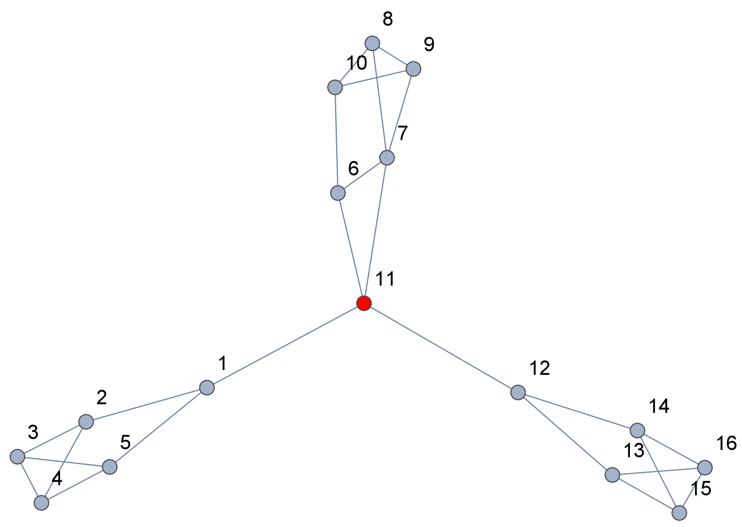


Figure 2: Example of windmill network, player 11 is the hub player

Figure 2 shows an example of a windmill network where there are three social groups (players 1-5, players 6-10 and players 12-16) and the groups are connected with each other through player 11, who is the hub player of this network.

Proposition 5 *In a windmill network if there is no social group that is bigger in size than the rest of the social groups combined, then the hub player is the (strong) Condorcet winner in the network.*

Proof Let us suppose that h is the hub player that connects the social groups $G_1, G_2, \ldots, G_\Lambda$ with each other. This means that any voter $i \in N \setminus G_\lambda$ inherits the preference ordering of h over any candidates from G_λ. Consequently in a 2-candidate election between h and any $j \in G_\lambda$ the hub player h has the support of all the voters $i \in N \setminus G_\lambda$. As no social group is bigger than the rest combined this means that the $N \setminus G_\lambda + 1$ support of h is always bigger than the support of any possible challenger. Since h cannot be beaten in a 2-candidate election by any challenger, the hub player is a (strong) Condorcet winner in a windmill network. □

3.3 Social quilts

The class of social quilt networks was introduced by Jackson et al. [2012]. In this paper the authors claim that the social quilt is an efficient social structure to maintain favor exchanges in the society by reduce the incentives to renegotiate and to avoid that the whole society collapse if someone fails to provide a favor (robustness). It is interesting to study the network class of social quilts in this paper because the interpersonal links in my model (in matrix C) can be interpreted as favor exchanges in the society. Although in my model I exclude the possibility of not providing a favor and deviating in the payoff phase of the voting game, it is good to know that this class of networks would be stable even if I relaxed that constraint in my model.

Definition 4 *An* ***m-type social quilt*** *is a union of small, completely connected subnetworks (cliques) of size m that are laced together in a tree-like pattern (without cycles) by citizens who belong to different (at most two) cliques simultaneously.*

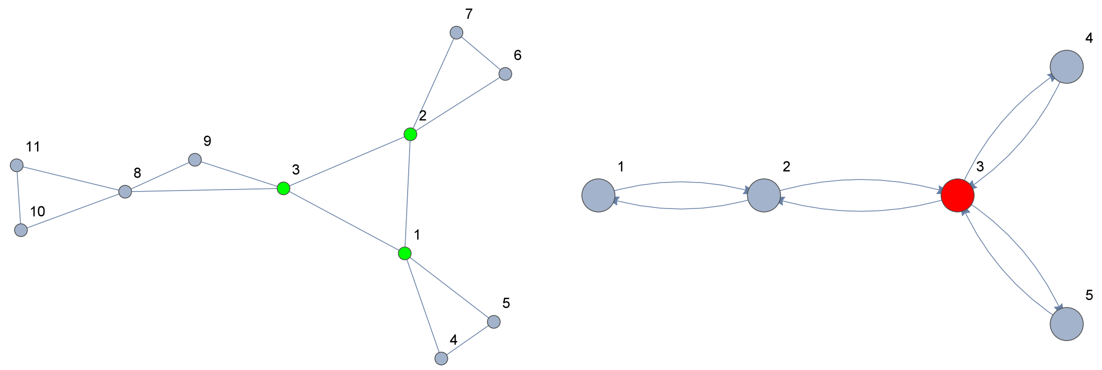


Figure 3: Example of social quilt and its tree representation

According to my definition of the class of social quilts there is a tree structure behind every social quilt.

Definition 5 C_T *is the* ***tree representation of the social quilt network*** C *if every clique in* C *appears as a node in* C_T *and every pair of cliques that share a member in* C *are connected in* C_T*.*

From Demange [1982] we know that if the nodes have single peaked preferences on a tree network there are one or two Condorcet winners in the network. I introduce the notion of winner clique for the cliques in a social quilt network whose corresponding node would be a Condorcet winner in the tree representation of the network if the nodes had single peaked preferences.

Definition 6 *W is a* ***winner clique*** *in a social quilt network C if the node w corresponds to W in C_T, and w would be a Condorcet winner in a network identical to C_T, where the nodes have single peaked preferences.*

Proposition 6 shows what role the winner cliques play in the network:

Proposition 6 *In the social quilt C*

1. *if there are two winner cliques (W_1 and W_2), the Condorcet winner always exists in the network and it is the citizen that belongs to both cliques, and*
2. *if there is one winner clique (W) in the quilt and the Condorcet winner exists, it is the member of the winning clique.*

Proof, first part Let S_1 be the set of nodes in C_T that are closer to w_1 than to w_2 and S_2 the set of nodes that are closer to w_2 than to w_1. Since w_1 and w_2 would be Condorcet winners on C_T if the nodes had single peaked preferences, $S_1 = S_2$ and $S_1 \cup S_2 = C_T$. Let S_1' and S_2' denote the set of citizens in C who belong to S_1 and S_2 respectively. We know that $S_1' = S_2'$ and $S_1' \cup S_2' = C$ and that the two sets has one element in common $i \equiv S_1' \cap S_2'$. From Lemma 2 we know that all the voters $S_1' \setminus i$ inherit the preference ordering of i over any set candidates $\{j_1, j_2, j_3, \ldots\} \subset S_2'$ and since i prefers herself to any other candidate in S_2', all the voters in $S_1' \setminus i$ support i against any other candidate from S_2'. As I stated before $S_1' = S_2'$ so $S_1' > (S_2' \setminus i)$, that means that candidate i beats every other candidate from S_2' in a 2-candidate contest. Changing the roles of S_1' and S_2' it is easy to see that i beats every other candidate from S_1' as well in a 2-candidate contest, so i is a Condorcet winner.

Proof, second part Suppose that the Condorcet winner w exists, but it is not in the winner clique ($w \in \tilde{W}$ and $\tilde{W} \neq W$). In C_T there is a W to $\tilde{W}$ path and V, one neighbor of W is on the path[10]. Let S_1 be the set of nodes in the tree representation C_T that are connected to $\tilde{W}$ through W. Since W is the only winner clique, $S_1 \cup \{W\} > C_T/2$. If S_1' is the set of voters in C who belong to the cliques in set S_1 then $S_1' \cup W > C/2$. As the cliques V and W are neighbors they share one node $i = V \cap W$ and all the nodes in $S_1' \cup W \setminus i$ are connected to $C \setminus (S_1' \cup W)$ through i. According to Lemma 2 the voters in $S_1' \cup W \setminus i$ inherit the preference ordering of i over the candidates from $C \setminus (S_1' \cup W)$ consequently they inherit the preference ordering of i over the candidate set $\{i, w\}$. By definition i prefers herself to w and

[10]Note that if W and $\tilde{W}$ are neighbors $V = \tilde{W}$.

so does every voter in $S'_1 \cup W \setminus i$, but since $S'_1 \cup W > C/2$ candidate w cannot win against i, this means that they cannot be a Condorcet winner. As we arrived to a contradiction the starting assumption cannot be true, if there is a Condorcet winner in the social quilt network (with one winning clique) then they are a member of the winning clique. □

This means that in case of the social quilt networks it is enough to study the winner clique(s) to find a Condorcet winner.

A candidate of the winner clique has to win the support of the clique in the first place to win the election.

3.4 Centrality and Condorcet winners

In this part I argue that the analysis of the social structure based on my model is complementary to the practice of using centrality measures to predict electoral outcome (as in Cruz et al. [2015] for example). The main advantage of working with centrality measures (e.g. betweenness or eigenvalue centrality) is that they can be easily defined for all players in any networks. From Cruz et al. [2015] we also know that high centrality is strongly correlated with electoral victory. However the analysis based on centrality measures does not take into account the network position of the candidates on the ballot relative to each other, only the centrality of each candidate.

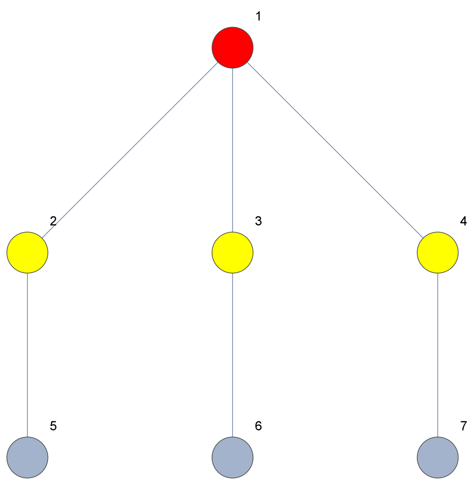


Figure 4: Peculiar ballot in Soc_1, player 1-4 runs

Let us consider for example the society Soc_1 from Figure 4 where player 1 to 4 runs for office - even if player 1 is the most central she will lose and players 2, 3 and 4 will tie on the first place. The approach based on centrality analysis is still very efficient in large samples when the influence of peculiar ballots (peculiar set of candidates) is relatively low[11]

Not only peculiar ballots but peculiar network structures can also confuse the analysis based on network centrality of the candidates. This can happen even when the electoral outcome is quite robust to the changes in the set of the running candidates, e.g. there is a strong Condorcet winner in the society who can beat any other candidate in a 2-candidate election. In this part I provide two examples of network structures where there is a strong Condorcet winner in the society, but she is not the most central player.

Let us consider first the society Soc_2 that is characterized by the social network from Figure 5 where the links are uniformly weighted. As there is no variation in link weights in the network the betweenness centrality of the nodes can be calculated by the formula designed for binary networks. The betweenness centrality of a node i is equal to the number of shortest paths from all nodes to all others that pass through node i. The vector of betweenness centralities in the society Soc_2 is $(0, 7, 12, 15, 16, 18, 0, 0, 0)$, so the player with the highest betweenness centrality is player 6. However player 5 is the (strong) Condorcet winner in the network, she can beat player 6 or any other player in a 2-candidate election.

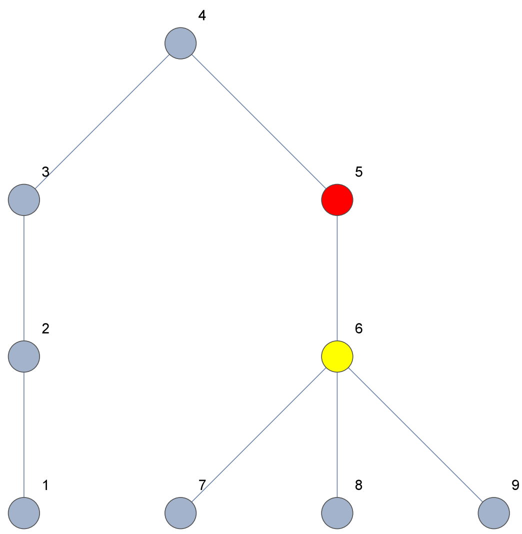


Figure 5: Betweenness centrality in Soc_2, example

[11]Cruz et al. [2015] uses electoral information from 15.000 villages and 709 municipalities.

For the second example let us consider the society Soc_3 characterized by the weighted adjacency matrix C (see also Figure 6):

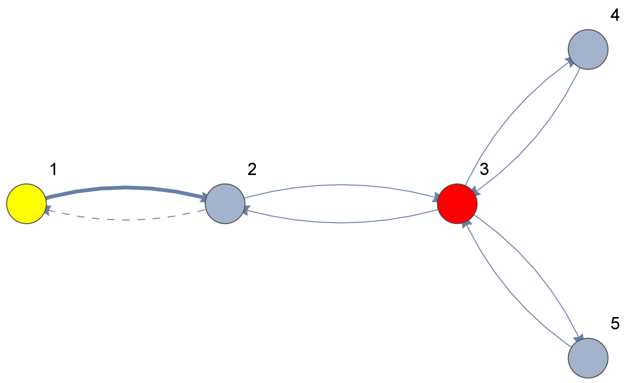


Figure 6: Eigenvector centrality in Soc_3, example

$$C = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 \\ 0.005 & 0 & 0.15 & 0 & 0 \\ 0 & 0.15 & 0 & 0.15 & 0.15 \\ 0 & 0 & 0.15 & 0 & 0 \\ 0 & 0 & 0.15 & 0 & 0 \end{pmatrix}.$$

The eigenvector that belongs to the leading eigenvalue and, by definition, determines the eigenvector centrality of the nodes in the society Soc_3 is $(0.29, 0.15, 0.26, 0.15, 0.15)$, so the player with highest eigenvector centrality is player 1. On the other hand player 3 is the Condorcet winner in the society, so she can beat player 1 or any other player in a 2-candidate election.

3.5 Application: Rise of the Medici

In this part I give an example how the model works in a real life situation. I use the Rise of the Medici family to show the predictive power of the model. This case is well documented by Padgett and Ansell [1993], and it has all the key elements of my model. The place is the Florentine city-state; the time is the early 15th century; the electorate is the Florentine nobility that forms a social network: the noble families are linked by marriage, business ventures, friendship, *etc.*; the election situation is the standoff of 1433 when the Florentine nobility voted with their feet about the future of the city-state: the supporters and the opponents of the Medici family gathered with their arms on the Piazza della Signoria to put an end to a long power struggle – the standoff did not escalated to fight since the Medici opponents lost the "vote", they were outnumbered.

To solve the model first I need to build and calibrate the social network. Unfortunately there is no available data of all the 215 noble families that are mentioned in Padgett and Ansell [1993] so I can only use data of a sample of 16 families[12]. This data set provides information both on the marriage and the business connections of the 16 Florentine noble families.

First thing that is worth to mention about the calibration of the model is that in this paper I follow Padgett and Ansell [1993] and take the family as the decision making actor of the society (not the individual). This is in line with the reality in Florence in the 15th century, where the patriarch of each family had a high influence on the marital, business and political decisions of the household. The other important thing to note is that there is anecdotal evidence in Padgett and Ansell [1993] that the marriage and business links between families played the role that the model predicts: linked families helped each other and provided indirect influence to each other in the same time. For example by coordinated voting Cosimo de' Medici managed to get his trusted friends elected to the key political positions of the city-state, who in turn paved the way for the Medici takeover. However the marital and the business ties had different importance: marriage meant a life-long binding and a symbolical merger of the two families, while business partnership in the 15th century Florence in most of the cases only meant one-time deals and guaranteeing for each other in credit agreements.

Based on the data set of the representative sample of the Florentine nobility I can calibrate the weighted adjacency matrix C of the city's elite. In Figure 7 marriage ties appear with gray, business links with red, and business and marriage ties with green[13]. To set the exact weights of each link I considered two things: (1) the sum of the link weights ($\sum_j c_{ij}$) should be less than 1 for every family; and (2) the weight of marriage links is higher than the weight of a business link.

I calibrated the network several times using different a and b weights for marriage

[12]They represent the elite of the nobility in terms of wealth and importance

[13]I dropped the Pucci family from the data set as they had neither marital nor business links to any of the families in the set.

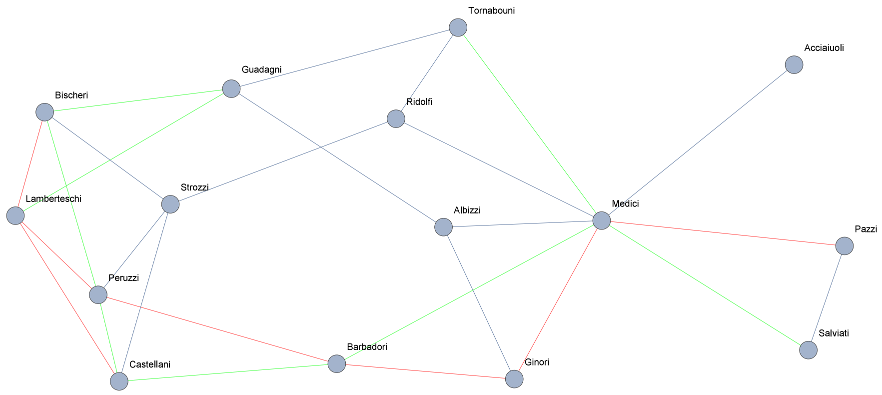


Figure 7: Florentine marriage network and business ties

and business respectively[14], taking into account that $\widehat{c_i} > 0$ for every family[15] and $a > b$. The result is always the same: the Medici family is Condorcet winner in the network, they can win a 2-candidate contest against any challenger with the majority of the votes.

To see that the Medici family is the Condorcet winner in the network let us consider the potential challengers one by one. The families Acciaiuoli, Salviati and Pazzi are connected to the rest of the network by the Medici family, so the rest of the network inherits the preferences of the Medici in elections where they are involved (and every family prefers itself to other families). As they can be ruled out as potential challengers of the Medici rule, their second best option is the victory of the Medici as they are too fare from the rest of the families. The situation is not so clear with the block on the left side of Figure 7 consisted by the Bischeri, Castellani, Lamberteschi, Peruzzi and Strozzi families. In fact these families were the backbone of the oligarchic rule that characterized Florence in the beginning of the 15th century and these were the families that opposed fiercely the rise of the Medici family. However, the oligarchs did not have the numbers to prevent the success of the Cosimo de' Medici. For example, in my model if the parameters are set $a = 0.1$, $b = 0.07$ and $\beta = 0.01$[16] and Medici runs against the Strozzi family the Medici wins the vote 10-5. No closely linked family (from the center of the graph) can successfully

[14]I was assuming additively separable relationship between the two kind of ties, so I used $a + b$ weight if a pair of families are connected both by business and marriage.

[15]The family with the highest number of outlinks is the Medici family, for them the inequality has the following form: $1 - 6a - 5b > 0$.

[16]The result is robust to the changes in β as much as to the changes in a and b.

challenge the Medici family either. For example the Albizzi family seems to be in a position to potentially divide and rule between the oligarchs on the left side of Figure 7 and the Medici supporters on the right, but when they run against the Medici they lose 10-5 since they cannot secure the full support of the oligarch block.

These model predictions are very much in line with the actual happening of the 1430s. First the traditional oligarch rule was challenged when Cosimo de' Medici managed to win the seats of the city council for his supporters, then Rinaldo Albizzi, following his own political ambition, led a conspiracy against the Medici family (wanted to purge the city council from the Medici supporters by force) but could not secure the support of Palla Strozzi, the richest man of Florence for his maneuver and eventually had to retreat. Figure 8 shows how Padgett and Ansell [1993] classified the families according to their alliances: the opponents of the Medici are black, the supporters are yellow and the families with split loyalties (brothers or father-son on different sides) are orange. We can see that the model predicts quite well which family supported whom during the power struggle: the left block is entirely black, they were the main opponents of the Medici rule, the right block is yellow and the Barbadori and Albizzi families on the frontier of the two competing blocks have split loyalties. The split loyalty of the Salviati family is a bit of a surprise, as it cannot be explained by the ties I have data on. However it can well be the case that this occurs because this network contains only a sample of the families and only two types of ties that link the families – probably this calibration misses actors or ties that shaped the political decision of one member of the Salviati family.

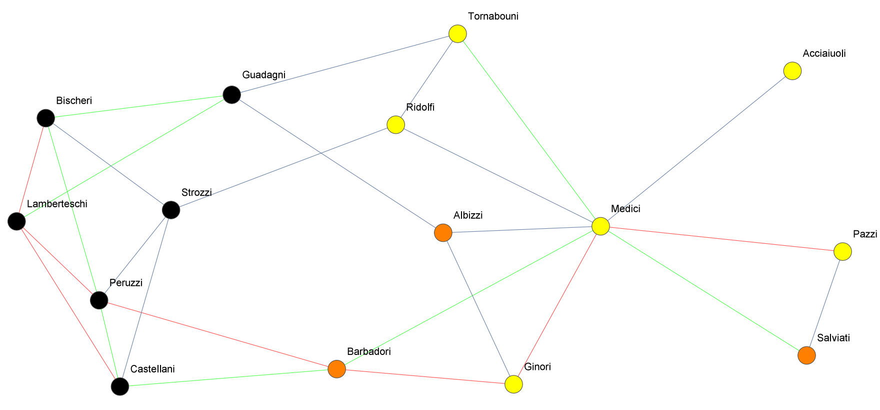


Figure 8: Florentine marriage network, business ties and alliances in the 1430s

4 Conclusion

This paper has developed a model of electoral competition where the preferences in the electorate are not based on general political ideology: the voting decision of the citizens is entirely based on the interpersonal relationships in the society. The collection of these interpersonal relations is represented as a social network. In this framework I study the result of a modified version of the three stage citizen-candidate voting game. I show that the voting game has a (trivial) political equilibrium in every network for some parameters values. Then I show that the model guarantees single peaked preferences on the line and the tree network, and that the tree representation of the social quilt network is a useful tool to identify the Condorcet winner. Finally I provide a real life example how the model can be used to reproduce real political outcome by calibrating the model and finding the Condorcet winner in the marriage and business network of the Florentine elite of the 15th century.

I believe that there are further questions in this topic that can be studied. As political importance seems to be related to the citizen's position on the social network, it would be natural curiosity to ask how did the social network develop into a given architecture and how did the individual's social decisions (forming new links or destroying old ones) affect this development. One possible way to answer that question is to endogenize the network formation, introducing a new stage in the voting game, when the self-selected candidates can revise their (endowment of) social connections and offer new links (or destroy existing ones) in order to maximize their probabilities of winning the election.

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