Various stable matching concepts

Tamás Fleiner   Zsuzsanna Jankó

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Stable marriages

Gale and Shapley (1962)
There are n men and m women, each of them having a preference order on the members of the other gender. We call a marriage scheme *stable* if there is no *blocking pair*: a man and women that mutually prefer each other to their own partners (or he/she is single).

Theorem (Gale-Shapley)

There always exists a stable matching, and it can be found with the deferred acceptance algorithm.

Optimality-pessimality

We call a stable matching S *male-optimal* it if is preferred by all men to any other stable matching: $S \geq_M S'$ for every stable matching S'. A stable matching S is *male-pessimal* if $S \leq_M S'$ for every stable matching S'.
Female-optimality and pessimality are defined similarly.

Theorem (Gale-Shapley)

The stable marriage scheme given by the Gale-Shapley algorithm is male-optimal and female-pessimal.

College admissions in Hungary

Given n applicants: $A_1, A_2, \dots, A_n$ and m colleges: $C_1, C_2, \dots C_m$.
Every applicant has a strict preference order over the colleges she applies to.
Every college assigns some score (an integer between 1 and M) to each of its applicants.
Moreover, each college C has a quota $q(C)$ on admissible applicants.
Each college has to declare a score limit. The score limit of college C_i is t_i.
The vector of declared score limits $(t_1, t_2, \dots, t_m)$ is called a *score vector*
Each applicant will become a student on her most preferred college where she has high enough score.

Properties of score vectors

Score vector $(t_1, t_2, \ldots t_m)$ is *valid* if no college exceeds its quota with these score limits.

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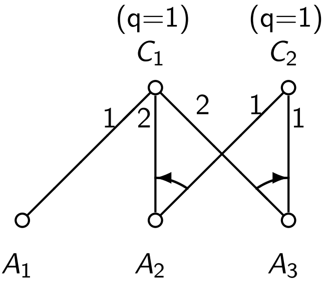
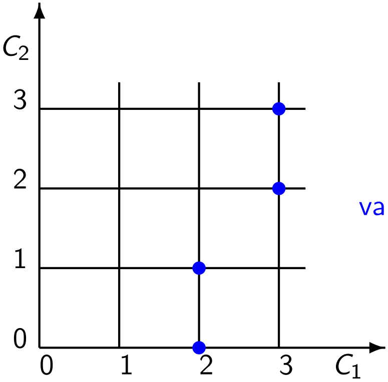
An student-college many-to-one matching is *score-stable* if it can be realized by a stable score vector.

Note that if applicants have different scores and the qouta is one for every college, then we are back at the stable marriage problem.

Example





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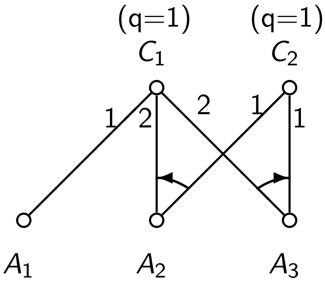

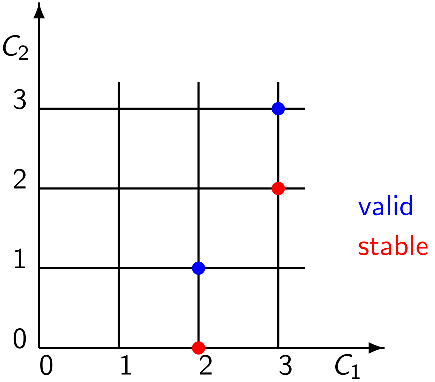

Score-decreasing algorithm

Theorem

For any finite set of applicants, colleges and set of applications, for arbitrary positive scores of the applications there always exists a stable score vector.

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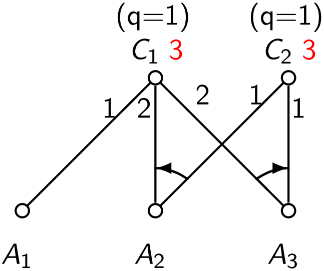
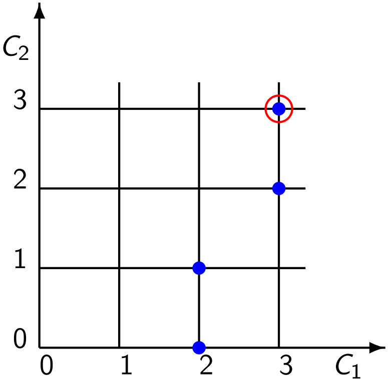
The are two natural algorithms to find a stable score vector:
1. **The score-decreasing algorithm**: colleges start on a valid score vector $\underline{t_C} := (M+1, \ldots, M+1)$ and they keep on decreasing their score limits by one at a time, if this results in another valid score vector. As soon as no college can decrease its score limit, the score vector is stable. Let $\underline{s_C}$ note the stable score vector we get.

Theorem

The score vector $\underline{s_C}$ maximal among all stable score vectors, and this assignment is applicant-pessimal.

Score-decreasing algorithm





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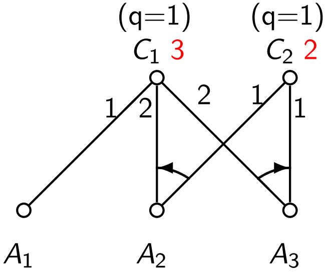

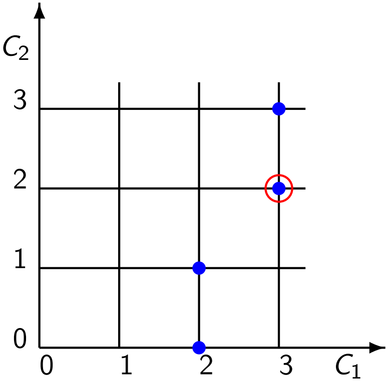

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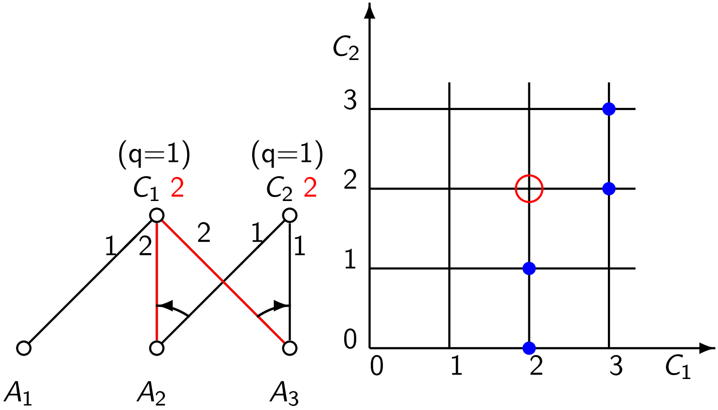

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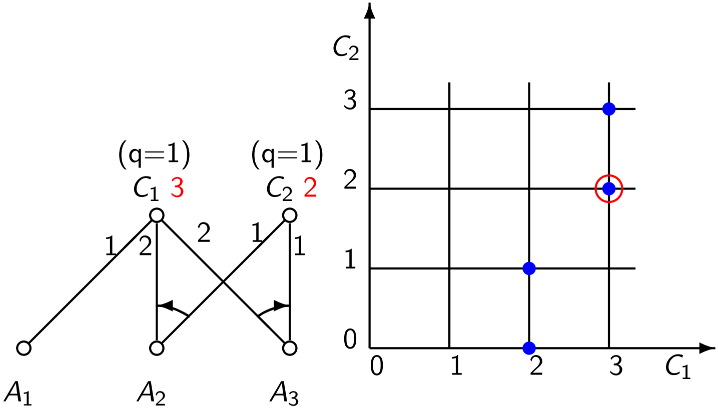

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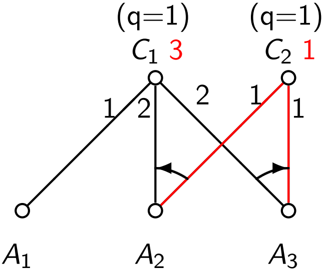

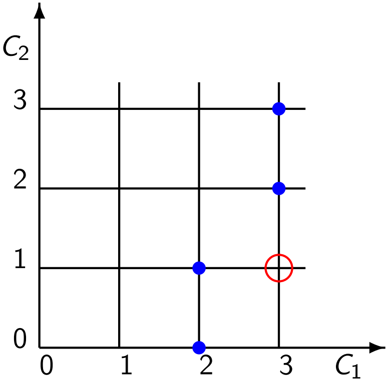

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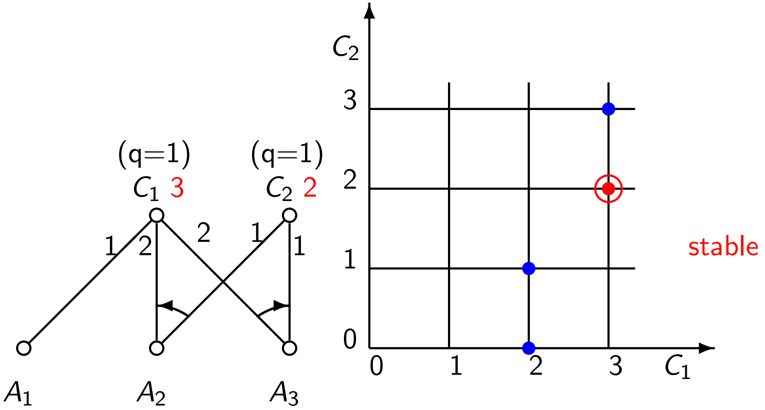


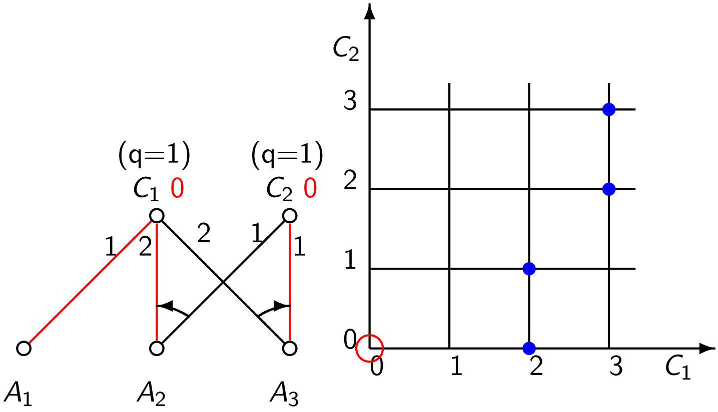
Score-increasing algorithm

2. **The score-increasing algorithm**: Colleges start with critical score vector $\underline{t_A} = (0, \ldots, 0))$ and keep on raising there score limits by one, if they receive more students than their quota. As soon as the score vector becomes valid, the score vector is also stable. Let $\underline{s_A}$ the stable score vector the score-increasing algorithm outputs.

Theorem

Score vector $\underline{s_A}$ is the minimum of all stable score vectors. Additionally it is applicant-optimal.

score-increasing algorithm



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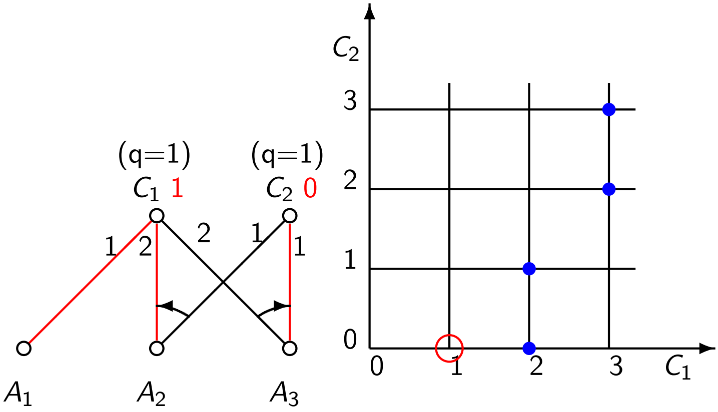

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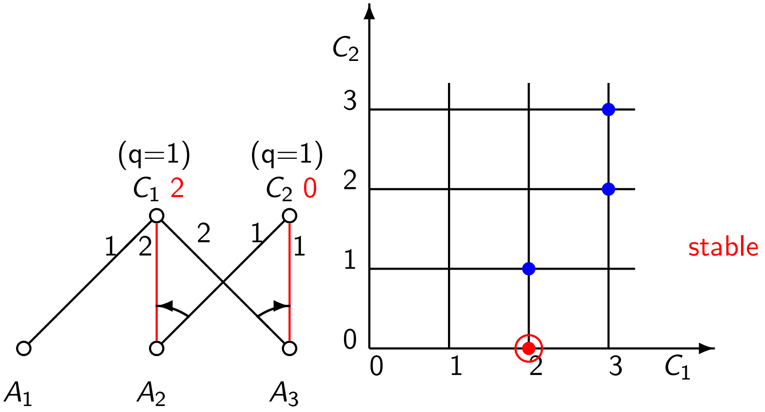

Choice functions and properties

Choice function: $\mathcal{F} : 2^E \to 2^E$ s.t. $\mathcal{F}(A) \subseteq A \quad \forall A \subseteq E$.

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We usually assume this proprety.

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A choice function $\mathcal{F}$ is **linear order based** (LOB) if it can be defined by a strict preference order over all subsets of E, such that $\mathcal{F}(A)$ is best subset of A according to this order.
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Path-independence

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Lemma (Fleiner)

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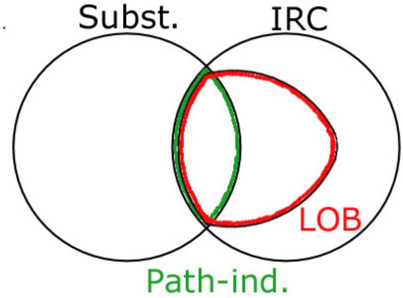
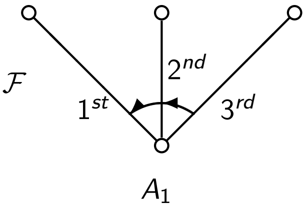


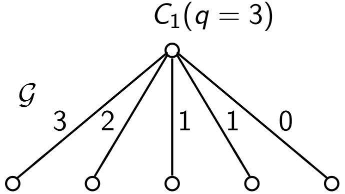
Two-sided market

A *two-sided market* can be represented as a bipartite graph. On one side, the applicants have a choice function $\mathcal{F}$ over the set of contracts and the on the other side the colleges have a choice function $\mathcal{G}$ over the contracts.
Let E be the set of all possible contracts.

Choice functions for college admissions

For subset $X \subseteq E$ of applications $\mathcal{F}(X)$ denotes the set of most preferred applications of each applicant.
Similarly, $\mathcal{G}(X)$ denotes the set of applications that colleges would choose. From a given set of applicants, they choose the most possible applicants by giving a score limit, not exceeding their quota.
Choice function $\mathcal{F}$ of the applicants is IRC, but $\mathcal{G}$ for the colleges is not. $\mathcal{F}$ and $\mathcal{G}$ are both substitutable.
Example:





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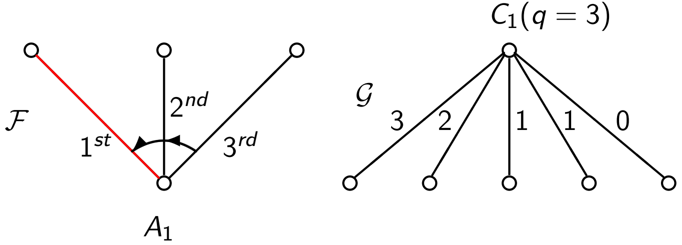

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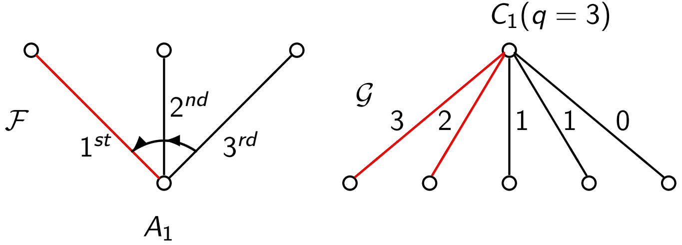

Pairwise stability

A contract-set S of E is called pairwise stable (or *dominating stable*), if

1. $\mathcal{F}(S) = \mathcal{G}(S) = S$ and
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This is a natural generalization of the original stable marriages.

Group-stability

Hatfield-Milgrom (2005) used the following concept (for many-to-one matchings), we will name it *group-stable*:
A set of contracts $S \subseteq E$ is *group-stable* if

1. $\mathcal{F}(S) = \mathcal{G}(S) = S$ and
2. there exists no college h and set of contracts $X' \neq \mathcal{G}_h(S)$ such that $X' = \mathcal{G}_h(S \cup X') \subseteq \mathcal{F}(S \cup X')$.

Lemma

If $\mathcal{F}$ and $\mathcal{G}$ are substitutable, and the market contains at most one contract between a given college and student, then group-stability and pairwise stability are equivalent.

3-stability

Subset S of E is 3-stable, if

1. $\mathcal{F}(S) = \mathcal{G}(S) = S$ and
2. there exists subsets A and B of E, such that $\mathcal{F}(A) = S = \mathcal{G}(B)$ and $A \cup B = E$, $A \cap B = S$.

Pair (A, B) with this property is called an *3-stable pair*, and S is an *3-stable core*.

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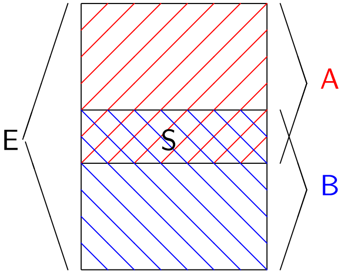

Score-stability versus 3-stability

For a set of applications S, score-stablitity is not equivalent with 3-stability.

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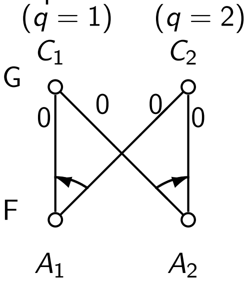

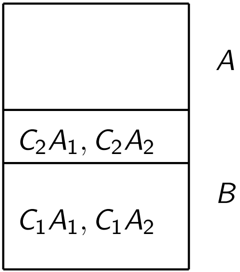

Let $A = S = \{C_2A_1, C_2A_2\}$ and $B = E$ therefore $\mathcal{F}(A) = \mathcal{G}(B) = S, A \cup B = E, A \cap B = S$. So S is 3-stable, and it can be realized with score vector $(1, 0)$. Although, $(1, 0)$ is not score-stable, if C_1 lowers its limit to $(0, 0)$, the admission changes to C_1A_1, C_2A_2 which is still valid.

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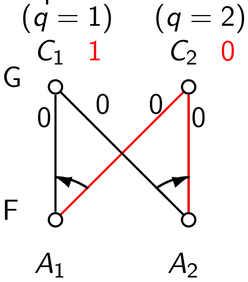

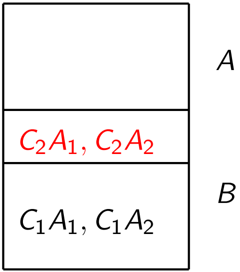

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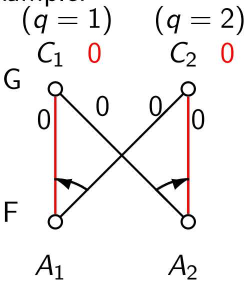
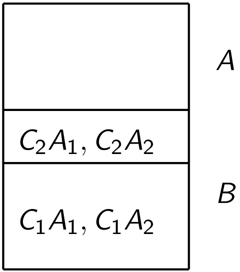
Example:





Let $A = S = \{C_2A_1, C_2A_2\}$ and $B = E$ therefore $\mathcal{F}(A) = \mathcal{G}(B) = S, A \cup B = E, A \cap B = S$. So S is 3-stable, and it can be realized with score vector $(1, 0)$. Although, $(1, 0)$ is not score-stable, if C_1 lowers its limit to $(0, 0)$, the admission changes to C_1A_1, C_2A_2 which is still valid.

Determinants

We say $\mathcal{D} : 2^E \to 2^E$ is a *determinant* of choice function $\mathcal{F}$ if $\mathcal{F}(A) = A \cap \mathcal{D}(A)$ for every $A \subseteq E$.

Lemma

Choice function $\mathcal{F} : 2^E \to 2^E$ is substitutable if and only if there exists an antitone determinant $\mathcal{D}$ of $\mathcal{F}$.

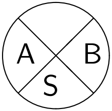
For every substitutable F, there is a *canonical determinant*, which is minimal among all possible antitone determinant of $\mathcal{F}$.

$$\mathcal{D}_{\mathcal{F}}(A) := \{e \in E : e \in \mathcal{F}(A \cup \{e\})\}$$

4-stability

Subset S of E is 4-stable, if

1. $\mathcal{F}(S) = \mathcal{G}(S) = S$ and
2. there exists subsets A and B of E, such that $\mathcal{F}(A) = S = \mathcal{G}(B)$ and $A \cap B = S$, $\mathcal{D}_F(A) = B$, $\mathcal{D}_G(B) = A$

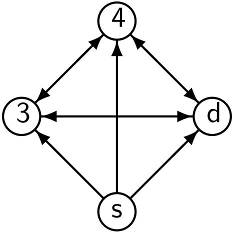


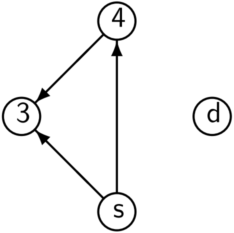
Connection between stability concepts

Theorem

If F and G are substitutable and IRC, 3-part, 4-part and dominating stability are equivalent.

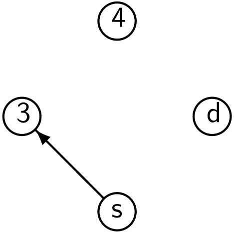
both F, G are IRC





one side is IRC

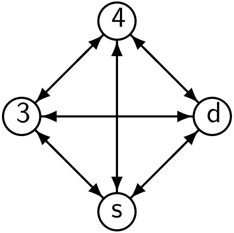
F and G may not be IRC

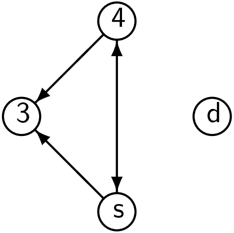


Connection between stability concepts 2

If there are no parallel contracts, i.e. there is only one possible contract between a college and an applicant.

both F, G are IRC





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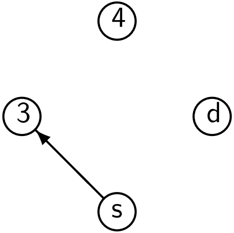

Partial order over stable marriages

There is a partial order over marriage schemes:
$S \geq_M S'$, if every men got at least as good wife in matching S as in S'. Similarly $S \geq_W S'$, if every woman got at least as good husband in S, compared to S'.

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Corollary:

Theorem (Conway)

The stable marriages form a distributive lattice for the partial order $\geq_M$.

Lattice properties

Theorem (Tarski's fixed point theorem)

Let $\mathcal{L}$ be complete lattice, and $f : \mathcal{L} \to \mathcal{L}$ be a monotone function on $\mathcal{L}$. Then $\mathcal{L}_f$ is a nonempty, complete lattice on the restricted partial order where $\mathcal{L}_f = \{x \in \mathcal{L} : f(x) = x\}$ is the set of fixed points of f.

Lattice of 3-stable cores

Given a choice function $\mathcal{F}$, define a partial order on contract sets: $S' \leq_{\mathcal{F}} S$ if $\mathcal{F}(S \cup S') = S$.

Theorem (Blair)

If $\mathcal{F}, \mathcal{G} : 2^E \to 2^E$ are substitutable, IRC choice functions, then the 3-stable cores form a lattice for partial order $\leq_{\mathcal{F}}$.

Recall that if both sides have IRC choice functions, 3-stability, 4-stability and dominating stability are equivalent, so all of them form a lattice.

Lattice properties of 4-stability

Theorem (Generalization of Blair's theorem)

If $\mathcal{F}$ and $\mathcal{G}$ are substitutable and $\mathcal{F}$ is IRC (G doesn't need to be IRC), the 4-stable sets form a nonempty complete lattice for partial order $\leq_{\mathcal{F}}$.

Lattice property of score-stability

If the market has no parallel contracts, 4-stability and score-stability are equivalent.

Theorem

If $\mathcal{F}$ and $\mathcal{G}$ are substitutable, and $\mathcal{F}$ is IRC, score-stable solutions form a lattice.

Conclusion

definition	always exists	lattice
dominating	no	no
group	no	no
3-stable	yes	yes
4-stable	yes	yes
score-stable	yes	yes

Usefulness of determinants

We have a choice funtion over a lattice $\mathcal{F} : \mathcal{L} \to \mathcal{L}$. For example every agent plays tennis with the others, and wants to allocate her free time. If she have 3 possible partners, and her free time all together is 1 hour, the choice function is $\mathcal{F} : [0,1]^3 \to [0,1]^3$. A choice function $\mathcal{F} : \mathcal{L} \to \mathcal{L}$ is substitutable if and only if there exists an antitone determinant $\mathcal{D}$ of $\mathcal{F}$.

Thank you for your attention!