Two applications of axiomatic ranking

László Csató
laszlo.csato@uni-corvinus.hu

Institute for Computer Science and Control, Hungarian Academy of Sciences (MTA SZTAKI)
Laboratory on Engineering and Management Intelligence, Research Group of Operations Research and Decision Systems

Corvinus University of Budapest (BCE)
Department of Operations Research and Actuarial Sciences

Budapest, Hungary

Workshop on Future Directions in Computational Social Choice
Hungarian Academy of Sciences, Országház u. 30., Budapest, Hungary
22 November 2016

Outline

1. **Ranking in Swiss-system chess team tournaments**
   - Problem, motivation
   - Connection to paired comparisons-based ranking
   - Axioms
   - Application

2. **University rankings on the basis of applicants' preferences**
   - Introduction
   - Derivation of revealed preferences
   - Connection to paired comparisons-based ranking
   - Axioms

3. **Summary**

Part I:
Ranking in Swiss-system chess team tournaments

Swiss system chess team tournaments

Characteristics

- Too many players to play a round-robin tournament (n is too large)
- A predetermined number of rounds ($c \ll n-1$) is organized
- Colour allocation does not count, no 'home advantage' (see later)

How to rank the teams on the basis of known results?

- Pairing algorithm is exogenous: matches between 'similar' teams
- Teams have different schedules

Measures of performance

- All matches are played on $2b$ boards: b players play with white and the other b players play with black in each team
- Board points: sum of points on the boards (win: 1, draw: 0.5, loss: 0)
- Match points: match outcome is decided by board points scored
  win: at least $b+0.5$ board points (win: 2, draw: 1, loss: 0)

Example: a match between two teams

Board number	Armenia	Hungary	Result
1	□ ARONIAN, Levon	■ BALOGH, Csaba	0.5 : 0.5
2	■ MOVSESIAN, Sergei	□ ALMASI, Zoltan	1 : 0
3	□ AKOPIAN, Vladimir	■ POLGAR, Judit	0.5 : 0.5
4	■ SARGISSIAN, Gabriel	□ BANUSZ, Tamas	0.5 : 0.5
Board points	2.5	1.5	
Match points	2	0	

Ranking in chess team tournaments

Official rankings

- Lexicographic order based on board or match points
- Board and match points do not depend on the strength of opponents
- Various tie-breaking rules: final result should be a strict total order

Notations

- **bp** is the vector of board points
- **mp** is the vector of match points

Board points ranking

The ranking derived from $\mathbf{bp}: i \succeq j \iff bp_i \geq bp_j$.

Match points ranking

The ranking derived from $\mathbf{mp}: i \succeq j \iff mp_i \geq mp_j$.

Match results, European Championship (EC) 2013

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	Match points
1		=		=	=	✔	✔	✔								✔									✔	=													14
2	=		✗	=	=		✔	✔	✔						✔						✔																		13
3		✔		✗						=	✔	✔	✔	✔										✗						✔									13
4	=	=	✔		✔				✗									✔			✔				=					✔									13
5	=	=		✗		=				=					✔				✔								✔					✔							12
6	✗				=		=	=	✔	=										✔								✔					✔						12
7	✗	✗				=				✔		=		=	✔		✔						✔																11
8	✗	✗				=					=	✔		✔	=									✔					✔										11
9		✗		✔		✗				=				=	=		✔				✔	✔																	11
10			=		=	=	✗		=							=		✔		✔											✔								11
11			✗					=													✔				✔		✔	✗		✗	✔							✔	11
12			✗				=	✗					=								=			✔	✔			✔						✔					11
13			✗									=			✗		=	=						✔		✔							✔			=			10
14			✗				=	✗	=								=													=	✔	✔						✔	10
15		✗			✗		✗	=	=				✔											✔			✔								✔				10
16	✗									=							=		✔	=	✗			=					✔			✔							10
17							✗		✗				=	=		=			✔			✔						=									✔		10
18				✗						✗			=						=		✗					✔					✔			✔		✔			10
19					✗											✗	✗	=		=		✔				✔			✔	=									9
20						✗				✗						=			=			=	✔	✗				✔				✔							9
21		✗		✗					✗		✗	=				✔		✔										✔							✔				9
22									✗								✗		✗	=			✔			✗		✔			✔						✔		9
23							✗													✗		✗					=				✗		✔	✔	✔		✔		9
24			✔					✗				✗	✗		✗	=				✔												=				✔			8
25	✗			=							✗	✗														=			✔	✔		✗		✔					8
26	=												✗					✗	✗			✔			=							✗				✔	✔		8
27					✗						✗				✗								=						✗				✔	=	✔			✔	8
28						✗					✔	✗					=			✗	✗	✗								✔		✔							7
29								✗								✗			✗						✗		✔				✗		✔	=		✔			7
30			✗	✗							✔			=					=						✗			✗			=			✔					7
31										✗	✗			✗				✗				✗	✔						✔	=								✔	7
32					✗									✗		✗				✗				=	✔	✔		✗							✔				7
33						✗							✗										✗				✗		✗						✔	=	✔	✔	7
34												✗						✗					✗		✗		=		=	✗							✔	✔	6
35															✗						✗		✗				✗					✗	✗			✔	✔	✔	6
36													=					✗						✗		✗			✗				=		✗		✗	✔	4
37																	✗					✗	✗			✗							✗	✗	✗	✔		✔	4
38											✗			✗													✗				✗		✗	✗	✗	✗	✗		0

Match results, EC 2013 (zoomed)

Rank	Team	1	2	3	4	5	...	Match points
1	Azerbaijan		=		=	=	...	**14**
2	France	=		✗	=	=	...	**13**
3	Russia		✔		✗		...	**13**
4	Armenia	=	=	✔		✔	...	**13**
5	Hungary	=	=		✗		...	**12**

The general mathematical model

Ranking problem (N, R, M)

- Set of alternatives: $N = \{X_1, X_2, \dots, X_n\}$
- Matches matrix M: symmetric, $m_{ii} = 0$ for all X_i
  $m_{ij} = m_{ji} \in \mathbb{N}$ is the number of comparisons between X_i and X_j
- Results matrix R: skew-symmetric, $r_{ii} = 0$ for all X_i
  $r_{ji} = -r_{ij}$ and $r_{ij} \in [-m_{ij}, m_{ij}]$

Ranking by scoring

- $\mathscr{R}^n$ is the set of ranking problems (N, R, M) such that $|N| = n$
- Scoring procedure $f : \mathscr{R}^n \to \mathbb{R}^n$
- Ranking: X_i is ranked weakly above $X_j \iff f_i(N, R, M) \geq f_j(N, R, M)$

Round-robin ranking problem

- Ranking problem (N, R, M) is *round-robin* if $m_{ij} = m$ for all $X_i, X_j \in N$
- $\mathscr{R}_R^n$ is the set of round-robin ranking problems such that $|N| = n$

Some scoring procedures

Notations

- $\mathbf{e} \in \mathbb{R}^n$ denotes the unit column vector: $e_i = 1$ for all $i = 1, 2, \dots, n$
- $L \in \mathbb{R}^{n \times n}$ is the Laplacian matrix of the comparison graph: $\ell_{ii} = \sum_{X_j \in N} m_{ij}$ and $\ell_{ij} = -m_{ij}$ for all $X_i, X_j \in N$
- $m = \max_{X_i, X_j \in N} m_{ij}$ is the maximal number of comparisons

Row sum ranking

- $\mathbf{s}(N, R, M) = R\mathbf{e}$, $s_i = \sum_{j \in N} r_{ij}$ for all $X_i \in N$

Least squares ranking

- The solution $\mathbf{q}$ of $L\mathbf{q} = \mathbf{s}$ and $\mathbf{e}^\top \mathbf{q} = 0$

Generalized row sum ranking

- The unique solution of $(I + \varepsilon L)\mathbf{x}(\varepsilon) = (1 + \varepsilon mn)\mathbf{s}$, $\varepsilon > 0$ is a parameter
- $\lim_{\varepsilon \to 0} \mathbf{x}(\varepsilon) = \mathbf{s}$ and $\lim_{\varepsilon \to \infty} \mathbf{x}(\varepsilon) = mn\mathbf{q}$

Modelling the tournament

Swiss-system chess team tournament as a ranking problem

- N consists of the teams of the competition
- Matches matrix M: $m_{ij} = 1$ if teams X_i and X_j have played against each other; $m_{ij} = 0$ otherwise
- r_{ij} depends on the match result (symmetric, draw: 0)

Results matrices

- *Board points based results matrix* R^{BP}: $r_{ij}^{BP} = (BP_{ij} - b)/b \in [-1,1]$
- *Match points based results matrix* R^{MP}: $r_{ij}^{MP} = MP_{ij} - 1 \in [-1,1]$

Lemma

Row sum ranking is equivalent to the official ranking without tie-breaking:

- $s_i(R^{BP}) \geq s_j(R^{BP}) \iff bp_i \geq bp_j$
- $s_i(R^{MP}) \geq s_j(R^{MP}) \iff mp_i \geq mp_j$

Theoretical properties I.

Axiom I: Score consistency

Scoring procedure $f : \mathscr{R}^n \to \mathbb{R}^n$ is called *score consistent* if $f_i(N,R,M) \geq f_j(N,R,M) \iff s_i(N,R,M) \geq s_j(N,R,M)$ for all $X_i, X_j \in N$ and round-robin ranking problem $(N,R,M) \in \mathscr{R}^n$.

Lemma

Row sum, generalized row sum and least squares methods are score consistent.

Corollary

Generalized row sum and least squares methods are equivalent to the official ranking without tie-breaking in round-robin tournaments:

- $x_i(\varepsilon)(R^{BP}) \geq x_j(\varepsilon)(R^{BP}) \iff q_i(R^{BP}) \geq q_j(R^{BP}) \iff bp_i \geq bp_j$
- $x_i(\varepsilon)(R^{MP}) \geq x_j(\varepsilon)(R^{MP}) \iff q_i(R^{MP}) \geq q_j(R^{MP}) \iff mp_i \geq mp_j$

Theoretical properties II.

Axiom II: Scale invariance

Let $(N, R, M), (N, kR, M) \in \mathscr{R}^n$ be two ranking problems such that $0 < k \leq \min_{X_i, X_j \in N} m_{ij}/|r_{ij}|$. Scoring procedure $f : \mathscr{R}^n \to \mathbb{R}^n$ is called *scale invariant* if $f_i(N, R, M) \geq f_j(N, R, M) \iff f_i(N, kR, M) \geq f_j(N, kR, M)$ for all $X_i, X_j \in N$.

Lemma

Row sum, generalized row sum and least squares methods are scale invariant.

Corollary

Let $(N, R, M) \in \mathscr{R}^n$ be a ranking problem, and $k \in (0, 1]$. Row sum, generalized row sum and least squares methods give the same ranking if they are applied on R^{BP} and kR^{BP} as well as on R^{MP} and kR^{MP}.

Theoretical properties III.

Notations

1. The *opponent set* of object X_i is $O_i = \{X_j : m_{ij} = 1\}$
2. Let $X_i, X_j \in N$ be two different objects and $g : O_i \leftrightarrow O_j$ be a one-to-one correspondence. Then $\mathfrak{g}$ is given by $X_{\mathfrak{g}(k)} = g(X_k)$.

Axiom III: Homogeneous treatment of opponents

Let $X_i, X_j \in N$ be two objects and $f : \mathscr{R}^n \to \mathbb{R}^n$ be a scoring procedure such that there exists a one-to-one mapping g from O_i onto O_j, where $f_k(N,R,M) = f_{\mathfrak{g}(k)}(N,R,M)$. f satisfies *homogeneous treatment of opponents* if $f_i(N,R,M) \geq f_j(N,R,M) \iff s_i(N,R,M) \geq s_j(N,R,M)$.

Lemma

Generalized row sum and least squares methods satisfy homogeneous treatment of opponents.

Message of the axioms

Score consistency

Generalized row sum and least squares ranking methods are equivalent to the official ranking without special tie-breaking rules if the tournament is round-robin (i.e. there are no constraints on the number of matches played).

Scale invariance

Generalized row sum and least squares ranking methods give a unique ranking on the basis of match points if wins are more valuable (have an arbitrary value in $(0, 1]$) than losses and draws correspond to an indifference relation.

Homogeneous treatment of opponents

The relative ranking of two teams depends only on their board/match points if they have played against opponents with the same strength.

Remark

Generalized row sum and least squares are iterative methods, they take the performance of opponents, opponents of opponents etc. into account.

Illustration: chess team European championships

Tournaments analysed

1. 18th European Chess Team Championship Open section (EC 2011)
   3-11 November 2011, Porto Carras, Greece
2. 19th European Chess Team Championship Open section (EC 2013)
   7-18 November 2013, Warsaw, Poland

Implementation

- Both tournaments: 38 participants, 9 rounds
- 171 matches are played from the possible $38 \times 37/2 = 703$ ($\approx 25\%$)
- Official ranking
- Least squares ranking(s)

Favourable results

- Comparison to the official ranking: more robust (between subsequent rounds), somewhat better in-sample fit, identical out-of-sample fit

Team	Official rank (0)	1	2	3	4	5	6	9	12	Cumulated change	**Least squares rank (∞)**
Azerbaijan	1	–	↓	–	–	–	–	–	–	↓	**2**
France	2	–	↑	–	–	–	–	–	–	↑	**1**
Russia	3	↓	–	–	–	–	–	–	–	↓	**4**
Armenia	4	↑	–	–	–	–	–	–	–	↑	**3**
Hungary	5	–	–	–	–	–	–	–	–	–	**5**
Georgia	6	–	–	–	–	–	–	–	–	–	**6**
Greece	7	–	–	–	–	–	↓	–	–	↓	**8**
Czech Rep.	8	↓	↓	–	–	–	–	–	–	↓↓	**10**
Ukraine	9	↑	–	–	–	–	↑	–	–	↑↑	**7**
England	10	–	↑	–	–	–	–	–	–	↑	**9**
Netherlands	11	↓ (6)	–	–	–	–	–	–	–	↓ (6)	**17**
Italy	12	↑	–	–	–	↓	–	–	–	–	**12**
Serbia	13	↓↓↓	↓↓	–	–	↓	–	–	–	↓ (6)	**19**
Romania	14	↓ (4)	↑↑	–	↑	–	–	–	–	↓	**15**
Belarus	15	↑↑↑	–	–	–	↑	–	–	–	↑ (4)	**11**
Poland	16	↑↑↑	–	–	–	↓	–	–	–	↑↑	**14**
Croatia	17	↑↑	–	–	↓	–	–	–	–	↑	**16**
Montenegro	18	↓	–	↓	–	–	–	–	↓	↓↓↓	**21**
Spain	19	↓↓	–	–	–	–	↓	–	–	↓↓↓	**22**
Germany	20	–	–	↑	–	↑	–	–	–	↑↑	**18**
Slovenia	21	↑ (7)	–	–	–	↑	–	–	–	↑ (8)	**13**
Poland Futures	22	↓↓	–	↓	–	↓	–	–	–	↓ (4)	**26**
Lithuania	23	↓↓	↓ (4)	–	–	–	↓	–	–	↓ (7)	**30**
Turkey	24	↑↑	–	–	–	–	↑	–	↑	↑ (4)	**20**
Bulgaria	25	↑↑	–	–	–	–	–	–	–	↑↑	**23**
Sweden	26	↓	–	↓	–	–	–	–	–	↓↓	**28**
Denmark	27	↓↓↓	↓	–	–	–	–	↓	–	↓ (5)	**32**
Israel	28	↑↑	↑	↑	–	–	–	–	–	↑ (4)	**24**
Iceland	29	↓↓↓	–	–	–	–	–	↑	–	↓↓	**31**
Austria	30	↑↑	↑↑	–	–	↑	–	–	–	↑ (5)	**25**
Poland Goldies	31	–	↑	–	–	–	↑	–	–	↑↑	**29**
Switzerland	32	↑↑↑	↑	↑	–	–	–	–	–	↑ (5)	**27**
Belgium	33	–	–	↓	–	–	–	–	–	↓	**34**
Finland	34	–	–	↑	–	–	–	–	–	↑	**33**
Norway	35	–	–	–	–	–	–	–	–	–	**35**
Scotland	36	–	–	–	–	–	–	–	–	–	**36**
FYR Macedonia	37	–	–	–	–	–	–	–	–	–	**37**
Wales	38	–	–	–	–	–	–	–	–	–	**38**

Part II:
University rankings on the basis of applicants' preferences

Hungarian higher education admission scheme

Main features

- Centralized system
- Students give an application for programmes
- Students have a (possibly different) score for each programme
- Each programme has a score limit determined by an algorithm
- Matching: a student *must* accept the first programme where his/her score is not lower than the limit

What is an application?

- It contains at most 5 programmes in a strict order
- State-funded and fee-paying form of two otherwise identical programmes count as one
- Example: 1st place – BA in International Business at Corvinus University of Budapest, Corvinus Business School (state-funded)

Preferences from applications I.

Preferences can be derived not only among programmes, but arbitrary objects (universities, *faculties*, courses etc.)

Assumptions

1. A higher ranked object is *preferred* to any lower ranked object
2. *No information* on preferences between two unranked objects
3. *No information* on preferences between a ranked object and an unranked object (Note: the length of the list is restricted)
4. If an object appears more than once in an application, only its best position counts: one student may have only one preference concerning a pair of objects

Preferences from applications II.

Original application

Rank	Faculty
1	SEAOK
2	DEFOK
3	SZTEAOK
4	SEAOK
5	DEAOK
6	SZTEAOK

Reduced application

Rank	Faculty
1	SEAOK
2	DEFOK
3	SZTEAOK
4	—
5	DEAOK
6	—

Revealed preferences

- SEAOK $\succ$ DEFOK
- SEAOK $\succ$ SZTEAOK
- SEAOK $\succ$ DEAOK
- DEFOK $\succ$ SZTEAOK
- DEFOK $\succ$ DEAOK
- SZTEAOK $\succ$ DEAOK

Example: the aggregated paired comparisons matrix of Dentistry and Medical faculties in 2013

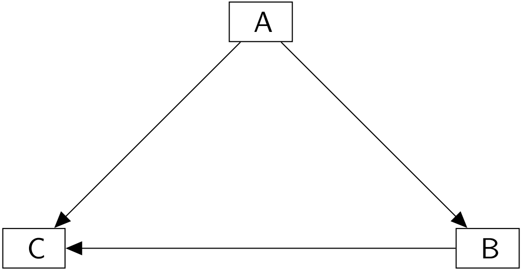
Faculty	Abbreviation	DA	DF	PA	PF	SA	SF	SZA	SZF	**Total**
DEAOK	DA	0	53	254	13	112	21	279	18	**750**
DEFOK	DF	99	0	24	60	16	24	25	53	**301**
PTEAOK	PA	271	18	0	39	110	24	285	19	**766**
PTEFOK	PF	28	59	92	0	15	24	27	53	**298**
SEAOK	SA	560	41	628	45	0	99	734	63	**2 170**
SEFOK	SF	51	155	78	145	129	0	54	173	**785**
SZTEAOK	SZA	467	25	474	27	92	18	0	40	**1 143**
SZTEFOK	SZF	33	109	45	100	14	22	92	0	**415**
Total		1 509	460	1 595	429	488	232	1 496	419	**6 628**

The mathematical model

- Aggregated paired comparisons matrix T: t_{ij} is the number of students preferring object X_i to object X_j
- Matches matrix M: $m_{ij} = t_{ij} + t_{ji} \in \mathbb{N}$ (symmetric)
- Results matrix R: $r_{ij} = t_{ij} - t_{ji} \in [-m_{ij}, m_{ij}]$ (skew-symmetric)

Graphical representation I.

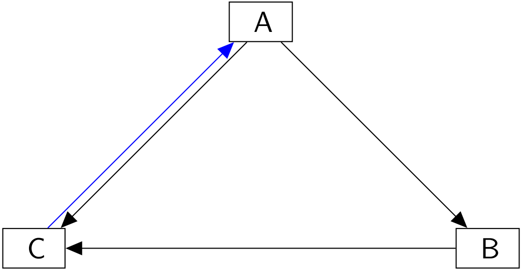
First student	
1st place	A
2nd place	B
3rd place	C



Graphical representation II.

First student	
1st place	A
2nd place	B
3rd place	C

Second student	
1st place	C
2nd place	A



Ranking methods

Nodes of a weighted, directed graph should be ranked.

Row sum: $\mathbf{s}(N, R, M)$

The difference of favourable and unfavourable preferences

Ratio: $\sum_j t_{ij} / \sum_j t_{ji}$

The ratio of favourable and unfavourable preferences

Generalized row sum: $\mathbf{x}(\varepsilon)(N, R, M)$
Least squares: $\mathbf{q}(N, R, M)$

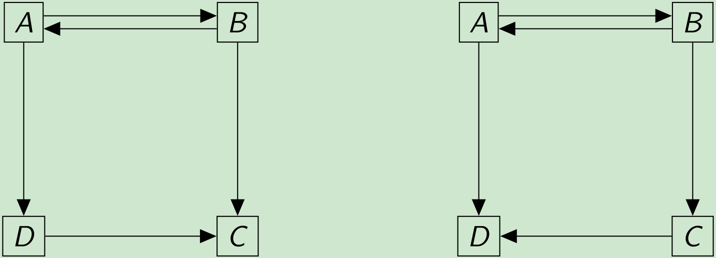
Solution of a system of linear equations, the quality of compared objects is taken into account

Theoretical properties I.

Axiom I: Independence of irrelevant matches (*IIM*)

Let $(N,T),(N,T') \in \mathscr{R}^n$ be two ranking problems and $X_k, X_\ell \in N$ be two different objects such that (N,T) and (N,T') are identical but $t'_{k\ell} \neq t_{k\ell}$. Scoring procedure $f: \mathscr{R}^n \to \mathbb{R}^n$ is called *independent of irrelevant matches* if $f_i(N,T) \geq f_j(N,T) \Rightarrow f_i(N,T') \geq f_j(N,T')$ for all $X_i, X_j \in N$.

The meaning of *IIM*



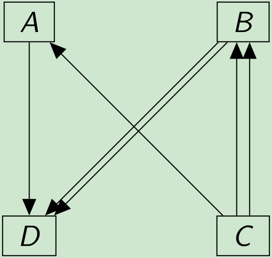
IIM implies $[A \succeq B$ in the first example$] \iff [A \succeq B$ in the second example$]$

Theoretical properties II.

Axiom II: Size invariance (SI)

Let $(N,T) \in \mathscr{R}^n$ be a ranking problem and $X_i, X_j \in N$ be two different objects such that $t_{jk} = \kappa t_{ik}$, $\kappa \in \mathbb{Z}^+$ for all $X_k \in N$. Scoring procedure $f : \mathscr{R}^n \to \mathbb{R}^n$ is called *size invariant* if $f_i(N,T) = f_j(N,T)$.

The meaning of SI



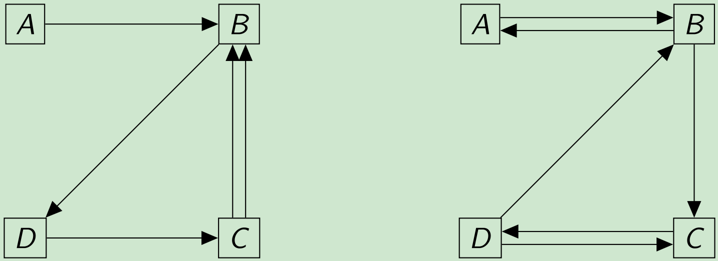
Size invariance implies $A \sim B$

Theoretical properties III.

Axiom III: Critical result preservation (*CRP*)

Let $(N,R,M) \in \mathscr{R}^n$ be a ranking problem and $X_i, X_j \in N$ be two different objects such that $m_{ik} = 0$ for all $X_k \in N$. Scoring procedure $f : \mathscr{R}^n \to \mathbb{R}^n$ satisfies *critical result preservation* if $f_i(N,R,M) \geq f_j(N,R,M) \iff a_{ij} \geq 0$.

The meaning of *CRP*



Critical result preservation implies $A \succ B$ in the first and $A \sim B$ in the second case

Axiomatic comparison of ranking methods

	$\mathbf{s}(N,R,M)$	Ratio	$\mathbf{x}(\varepsilon)(N,R,M)$	$\mathbf{q}(N,R,M)$
IIM	✔	✔	✗	✗
SI	✗	✔	✗	✔
CRP	✗	✗	✗	✔

Conclusions

Key points

- Ranking on the basis of paired comparisons between objects
- Two potential fields of applications
- Mathematical expression of reasonable requirements in both cases

Future research directions

- Refinement of the axiomatic approach: possibility/impossibility theorems, characterizations
- Further integration of axioms and applications

Thank you for your attention!