Ex-Ante Stable Lotteries

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Motivation

- Traditionally public school enrollment in neighborhood schools.
- Public School Choice gives students (or their parents) possibility to choose.
- Goal: promote diversity and equal access to good public schools, increase school quality through competition.
- large programs in NYC, Boston, New Orleans, etc.
- Mechanism Design approach starting with Abdulkadiroğlu and Sönmez, AER 2003.
- Design Goals: Fairness, Efficiency, Incentive Compatibility.

Priorities and Lotteries

- School choice is a problem of allocating indivisible objects to agents.
- Cannot use prices to allocate. Education should be free (for the students...). Thus we have to ration.
- Agent have different priorities for the objects. Priorities are thick. Many agents have the same priority for some objects.
  - Thick priorities are a generic problem in school choice.
  - e.g. Boston: priorities by walk-zones and siblings in the school.
- With indivisible objects, we may have to treat agents differently although they may have the same preferences and priority.
- However, we can try to restore fairness by using lotteries.

Literature

- Tie breaking lottery to run deterministic mechanism. (Erdil and Ergin, AER 2008, Abdulkadiroglu et al. AER 2009, Ashlagi et al. 2015)
  - Potential issue: Efficiency loss due to artificial constraints (Erdil, Ergin, AER 2008)
- Other Approach: Design lotteries from scratch satisfying some notion of ex-ante fairness, efficiency and incentive properties. (Kesten and Ünver, TE 2015, He et al. 2015)
- Designing fair lotteries is a challenge for mechanism design, on the other hand, randomness can be useful for evaluation (Abdulkadiroğlu et al. 2015)

Contribution

- Study ex-ante fair (stable) lotteries.
- Give a sense on how rich this class of lotteries is by giving bounds on the support.
- Show that they are close to deterministic in the sense that they have small support.
- Establish "constraint welfare theorems". Ex-ante fair and "efficient" lotteries can be decentralized by a pricing mechanism with priority specific pricing.
- Use graphical proof to do so.

Model

- N set of **agents**.
- M set of heterogeneous **object types**.
- Each $j \in M$ has $q_j \in \mathbb{N}$ copies.
- Assume $\sum_{j \in M} q_j = N$.
- Each i has strict **preferences** P_i over M. $P = (P_i)_{i \in N}$.
- Each j has **priority ranking** $\succeq_j$ over N (in general not strict).

Model (cont.)

- **Matching:** $\mu : N \to M$ such that for $j \in M$ we have $|\mu^{-1}(j)| = q_j$.
- **Allocation:** Lottery over matchings. Allocations can be written as $\Pi = (\pi_{ij}) \in \mathbb{R}^{N \times M}$ such that

$$0 \leq \pi_{ij} \leq 1, \quad \sum_{j \in M} \pi_{ij} = 1, \quad \sum_{i \in N} \pi_{ij} = q_j,$$

and vice versa.

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  and vice versa.
- Π is **ex-post stable** (ex-post fair), if it is a lottery over stable matchings.
- Agent i and object type j **ex-ante block** allocation Π if there is $i' \neq i$ with $\pi_{i'j} > 0$ and $i \succ_j i'$ and j' with $\pi_{ij'} > 0$ such that $j \, P_i \, j'$. Π is **ex-ante stable** (ex-ante fair) if it is not blocked by any pair i, j.

Ex-ante stability

Proposition (Roth et al. 1992, Kesten and Ünver 2015)

Each ex-ante stable lottery is ex-post stable.

- Not every ex-post stable lottery is ex-ante stable.

Ex-ante vs. ex-post stability

Suppose $q_1 = q_2 = q_3 = 1, q_4 = 2$.

P_1	P_2	P_3	P_4	P_5	$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
c	a	c	b	b	5	4, 5	1, 3	$\vdots$
a	d	d	d	a	1	$\vdots$	$\vdots$	
d	$\vdots$	$\vdots$	$\vdots$	$\vdots$	2			
$\vdots$					$\vdots$			

Take

$$\begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} = \begin{cases} \frac{1}{4} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ d & d & c & b & a \end{pmatrix} + \frac{1}{4} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & d & c & d & b \end{pmatrix} \\ + \frac{1}{4} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ c & d & d & b & a \end{pmatrix} + \frac{1}{4} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ c & a & d & d & b \end{pmatrix} \end{cases}$$

Ex-ante vs. ex-post stability

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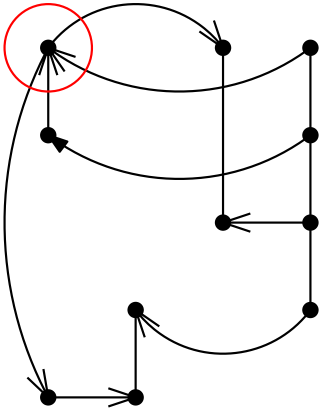
P_1	P_2	P_3	P_4	P_5	$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
c	a	c	b	b	5	4, 5	1, 3	$\vdots$
a	d	d	d	a	1	$\vdots$	$\vdots$	
d	$\vdots$	$\vdots$	$\vdots$	$\vdots$	2			
$\vdots$					$\vdots$			

Take

$$\begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} = \begin{cases} \frac{1}{4} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ d & d & c & b & a \end{pmatrix} + \frac{1}{4} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & d & c & d & b \end{pmatrix} \\ + \frac{1}{4} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ c & d & d & b & a \end{pmatrix} + \frac{1}{4} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ c & a & d & d & b \end{pmatrix} \end{cases}$$

A graphical representation of ex-ante stability[1]

$$\begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$



- Ex-ante stability means that there is no node with an incoming horizontal and vertical arc with origin in the support.

[1] Inspired by Balinski and Ratier, 1997

How rich is the class of ex-ante stable lotteries?

- We derive bounds on the size of the support of ex-ante stable lotteries.
- The **cut-off priority class** of j under Π is the lowest priority class $I_j(\Pi)$ with $i \in I_j(\Pi)$ such that $\pi_{ij} > 0$.
- The support of an ex-ante stable lottery is determined by the sizes of the cut-off classes.

Proposition

If Π is ex-ante stable, then

$$|supp(\Pi)| \leq n + \sum_{j \in M} |I_j(\Pi)|.$$

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Corollary

If priorities are strict and Π is ex-ante stable, then

$$|supp(\Pi)| \leq n + m.$$

In this case, each agent is matched to at most two object types. There is at most one copy of each object type that is matched to two agents. All other objects are matched deterministically.

Interpretation

Proposition

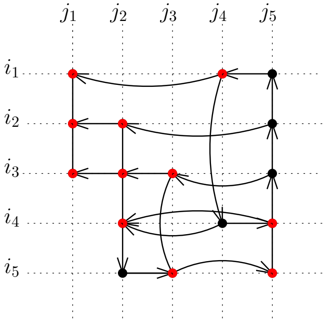
If Π *is ex-ante stable, then*

$$|supp(\Pi)| \leq n + \sum_{j \in M} |I_j(\Pi)|.$$

- Interpretation: The class of ex-ante stable lotteries is not much larger than the class of ex-post stable lotteries.
- Sophisticated lotteries cannot do much more than random tie breaking.

Proof by picture

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{2}{3} & 0 & \frac{1}{6} \end{pmatrix}$$



- We count red dots:

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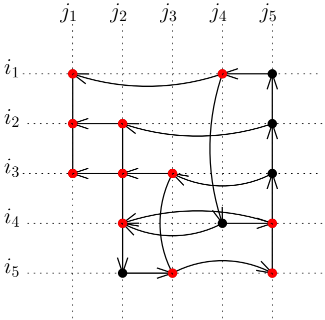
$$\begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{2}{3} & 0 & \frac{1}{6} \end{pmatrix}$$



- We count red dots:

$$|\mathrm{supp}(\Pi)| - n \leq |\{\text{red dots}\}| \leq \sum_{j \in M} |I_j(\Pi)| \quad \square$$

Pseudo-market mechanisms

- We change the set-up by introducing cardinal utility.
- $U = (U_i)_{i \in N}$ with $U_i \in \mathbb{R}_+^M$ a (vNM-)utility profile.
- U_i induces P_i such that

$$u_{ij} > u_{ij'} \Leftrightarrow j \, P_i \, j'$$

- Cardinal utility contains information not only about ranking but also about rate of substitution of probability shares.
- Moreover, each agent is endowed with a budget $b_i \in \mathbb{R}_+$.

Pseudo-market mechanism (He et al., 2015)

- A **pricing equilibrium with priority specific pricing** is a triple (C, p, Π) consisting of
  - $C = (C_j)_{j \in M}$ a set of cut-off indifference classes.
  - $p \in \mathbb{R}^M_+$ a vector of cut-off prices
  - an allocation Π

  such that defining

$$p^i_j = \begin{cases} \infty & \text{if } C_j \succ_j i, \\ p_j & \text{if } i \in C_j, \\ 0 & \text{if } i \succ_j C_j, \end{cases}$$

  in allocation Π, each agent maximizes expected utility subject to his budget constraint

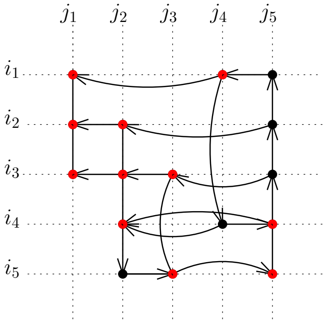
$$(\pi_{ij})_{j \in M} \in \operatorname{argmax} \left\{ \sum_{j \in M} u_{ij} \cdot \pi_{ij} \text{ subject to} \sum_{j \in M} p_{ij} \cdot \pi_{ij} \leq b_i \right\}.$$

U_1	U_2	U_3	U_4	U_5
$4\frac{1}{4}$	$4\frac{1}{4}$	$3\frac{1}{4}$	1	2
2	$3\frac{3}{4}$	$2\frac{3}{4}$	$3\frac{3}{4}$	3
1	2	$2\frac{1}{4}$	2	$3\frac{1}{4}$
4	1	1	3	1
3	3	2	$3\frac{1}{2}$	$3\frac{1}{2}$

$\succeq_1$	$\succeq_2$	$\succeq_3$	$\succeq_4$	$\succeq_5$
$1, 2, 3$	5	$3, 5$	4	1
$\vdots$	$2, 3, 4$	$\vdots$	1	2
	$\vdots$		$\vdots$	3
				$4, 5$
				$\vdots$

$$b = \left(\tfrac{11}{12}, \tfrac{2}{3}, \tfrac{5}{8}, \tfrac{1}{3}, \tfrac{1}{4}\right)$$

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{2}{3} & 0 & \frac{1}{6} \end{pmatrix}$$



- The red dots determine the cut-off classes.
- Cut-Off Prices: $p = (\frac{5}{4}, \frac{3}{4}, \frac{1}{4}, 1, \frac{2}{3})$.
- E.g. for agent 1:

$$\max 4\frac{1}{4} \cdot \pi_{11} + 4 \cdot \pi_{14} + 3 \cdot \pi_{15}$$
$$\frac{5}{4} \cdot \pi_{11} + \pi_{14} \leq \frac{11}{12}$$

Theorem (He et al. 2015)

For any utility profile U, priorities $\succeq$ and budgets b,

- *an equilibrium exists,*
- *there exists a selection from the equilibrium correspondence that is asymptotically incentive compatible.*

Proposition (He et al. 2015)

Each equilibrium allocation is ex-ante stable with respect to the induced ordinal preferences.

Proof.

Suppose i, j ex-ante block. Then $p_{ij} = 0$. $\square$

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- Natural question: Which ex-ante stable allocation can be decentralized in a pricing equilibrium?

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Proof.

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- Natural question: Which ex-ante stable allocation can be decentralized in a pricing equilibrium?
- Not all of them. For each utility profile inducing the preferences below, the allocation is not an equilibrium.

P_1	P_2
2	1
1	2

$\succeq_1$	$\succeq_2$
1, 2	1, 2

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

- Need to add some efficiency.

Improvement Cycles

- A **strong stable improvement cycle** is a sequence $(i_1, j_1), \ldots, (i_k, j_k)$ such that for each $1 \leq \ell \leq k$ (taking indices modulo k) the following holds:
  1. Agent i_ℓ is fractionally matched to j_ℓ under Π.
  2. Agent i_ℓ prefers $j_{\ell+1}$ to j_ℓ.
  3. Agents i_ℓ and $i_{\ell-1}$ have the same priority at j_ℓ.
- If there is a strong stable improvement cycle, we can achieve an Pareto improvement by reallocating probability shares within priority classes.

"Welfare Theorems"

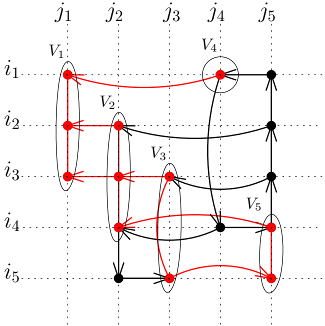
Theorem

Let Π be a random assignment that is obtained in an equilibrium with priority-specific pricing. Then Π has no strong stable improvement cycles.

Theorem

Let Π be ex-ante stable and free of strong stable improvement cycles with respect to P and $\succeq$. Then there exist budgets b, some utility representation U of P, cut-off prices p and cut-off classes C such that (C, p, Π) is an equilibrium with priority specific pricing.

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{2}{3} & 0 & \frac{1}{6} \end{pmatrix}$$



Future directions?

- What are the objectives?
  - Do school mechanisms improve school quality? (Hatfield, Kojima, Narita 2014)
  - Promote diversity, affirmative actions: Want to positively discriminate in favor of minorities. How to do it? Can we do it? (Kojima GEB 2012, Hafalir et al. TE 2015, Dogan JET 2016, Echenique and Yenmez AER 2015).
  - General Equilibrium effects (Avery and Pathag, 2015)
- Challenging from a design perspective. Many impossibilities e.g. Kojima, (GEB 2012).
- The design of the priorities/choice functions matter as much as the mechanism we use.
- "Choice and Matching" (Chambers and Yenmez, 2014)

Future Directions? (cont.)

- Empirics
  - We have two decades of school choice practice. What can we learn from data?
  - Welfare effects of centralized assignment (Abdulkadiroglu, Agarwal, Pathak, 2015)
  - Value added of education. (Abdulkadiroglu, Angrist, Pathak, ECTA 2014)
  - The use of lotteries can be useful for identification. Regression discontinuity design. (Abdulkadiroglu, Angrist, Narita, Pathak, 2016)

Future Directions in Matching

- New applications, e.g. the refugee match (Delacrétaz, Kominers, Teytelboym, 2016).
- Dynamic matching, Kidney exchange
  - Dynamics: The question when to match not just whom to match matters. (Ashlagi et al. 2016, Akbarpour et al. 2015, Anderson et al. 2014, Ünver, ReStud 2010)
  - Computational issues: Anderson et al. (PNAS 2015)
  - Dynamic Stability (Doval 2016, Kotowski and Kadam 2016)
- Stability under uncertainty (Liu et al. ECTA 2014)

Thank you! Questions?