

Popular matchings

Ágnes Cseh

Hungarian Academy of Sciences



Future Directions in ComSoC, 21 November 2016

Popular matchings

Ágnes Cseh

Hungarian Academy of Sciences



Future Directions in ComSoC, 21 November 2016

Outline

- ① Stable marriages
 - definition and algorithms
 - most important results
- ② Popular matchings
 - definition and algorithms
 - most important results and possible future directions
 - dominant matchings
- ③ Open questions

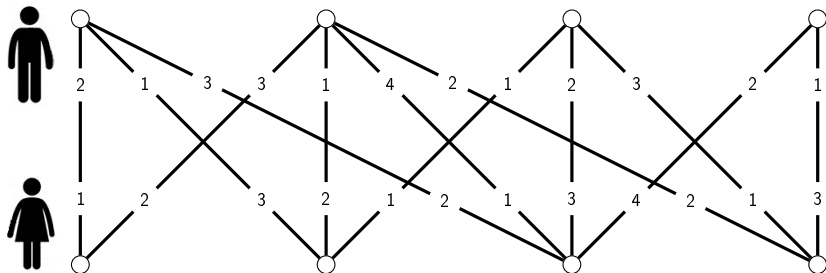
Definition



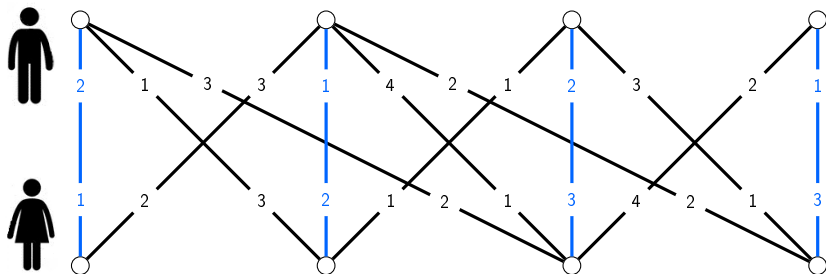
Definition



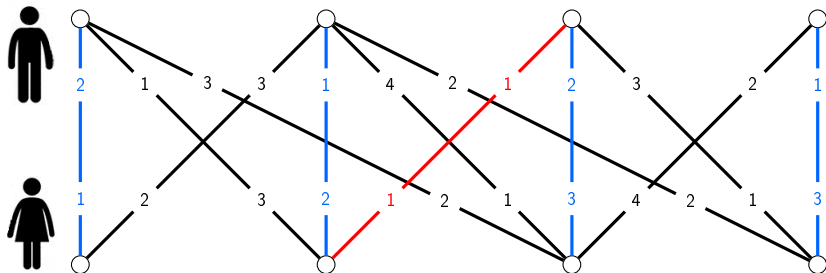
Definition



Definition



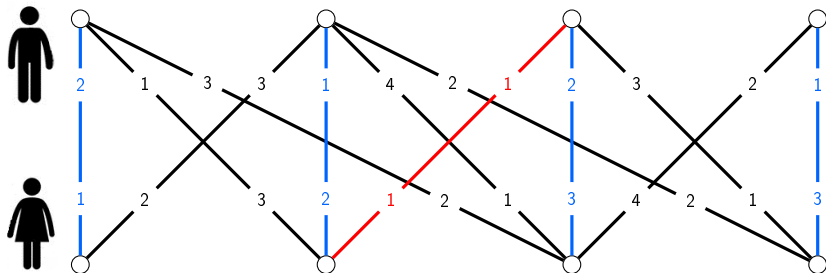
Definition



Definition

Edge mw is *blocking* if

Definition

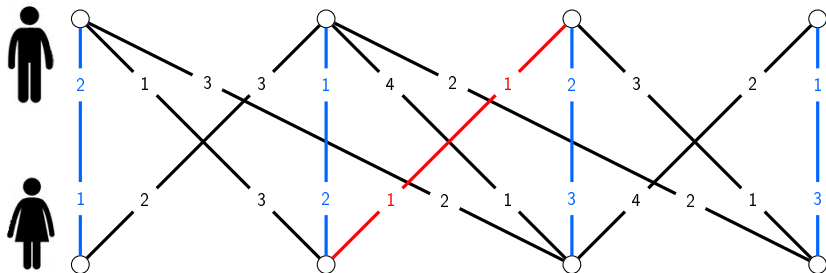


Definition

Edge mw is *blocking* if

- 1 it is not in the matching and

Definition

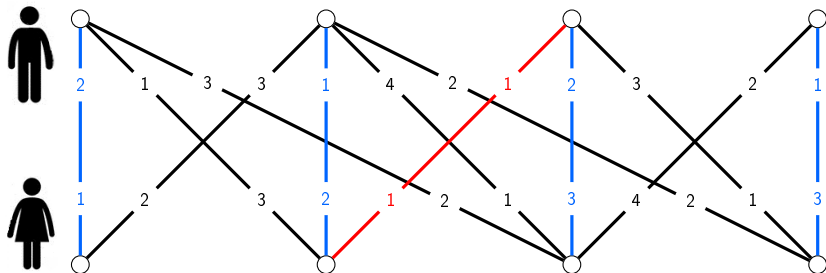


Definition

Edge mw is **blocking** if

- ① it is not in the matching and
- ② m prefers w to his wife or he is single and

Definition

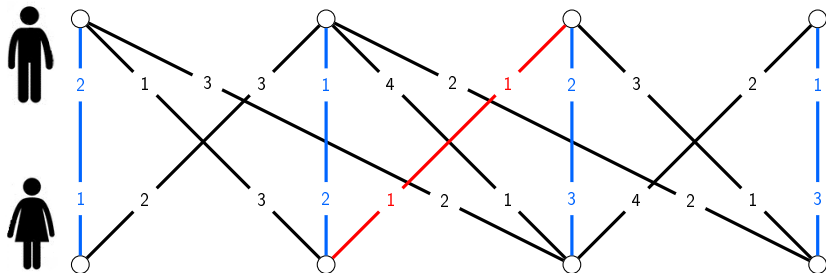


Definition

Edge mw is *blocking* if

- ① it is not in the matching and
- ② m prefers w to his wife or he is single and
- ③ w prefers m to her husband or she is single.

Definition



Definition

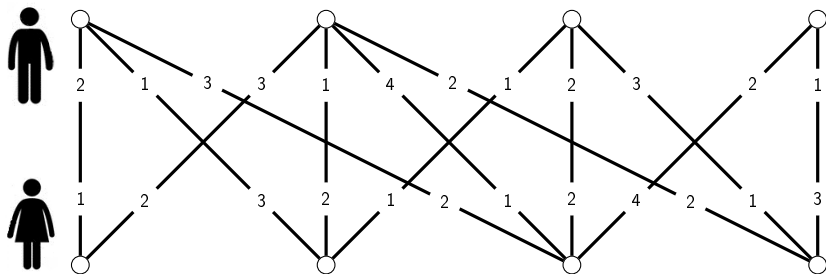
Edge mw is *blocking* if

- ① it is not in the matching and
- ② m prefers w to his wife or he is single and
- ③ w prefers m to her husband or she is single.

Theorem (Gale, Shapley 1962)

A stable matching always exists.

The Gale-Shapley algorithm

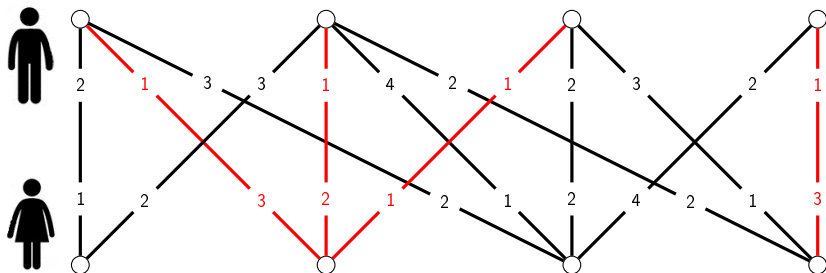


Definition

Edge mw is *blocking* if

- ① it is not in the matching and
- ② m prefers w to his wife or he is single and
- ③ w prefers m to her husband or she is single.

The Gale-Shapley algorithm



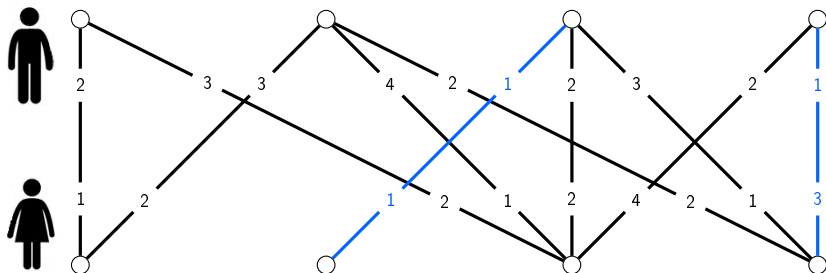
Definition

Edge mw is *blocking* if

- ① it is not in the matching and
- ② m prefers w to his wife or he is single and
- ③ w prefers m to her husband or she is single.



The Gale-Shapley algorithm



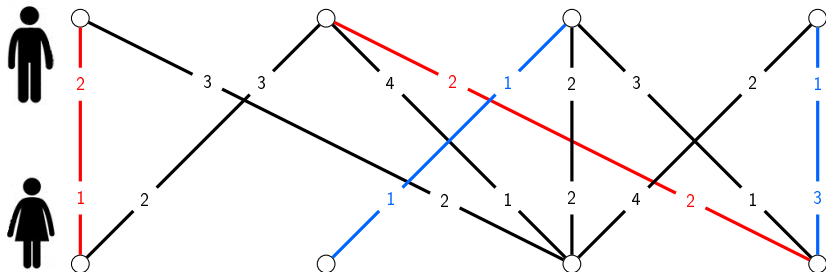
Definition

Edge mw is *blocking* if

- ① it is not in the matching and
- ② m prefers w to his wife or he is single and
- ③ w prefers m to her husband or she is single.



The Gale-Shapley algorithm



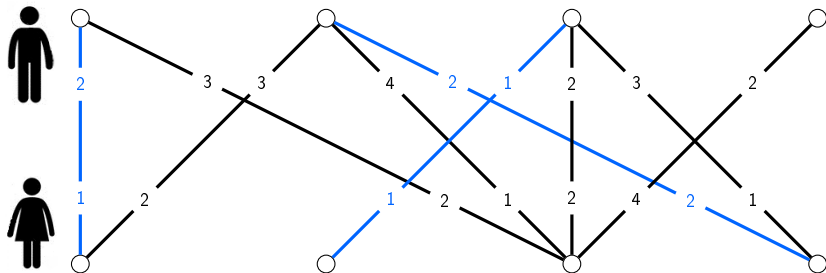
Definition

Edge mw is *blocking* if

- ① it is not in the matching and
- ② m prefers w to his wife or he is single and
- ③ w prefers m to her husband or she is single.



The Gale-Shapley algorithm



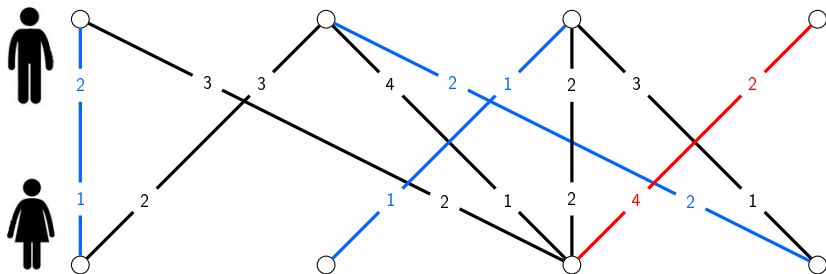
Definition

Edge mw is *blocking* if

- ① it is not in the matching and
- ② m prefers w to his wife or he is single and
- ③ w prefers m to her husband or she is single.



The Gale-Shapley algorithm



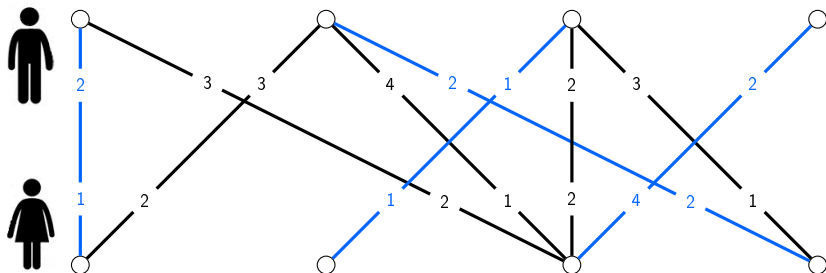
Definition

Edge mw is *blocking* if

- ① it is not in the matching and
- ② m prefers w to his wife or he is single and
- ③ w prefers m to her husband or she is single.



The Gale-Shapley algorithm



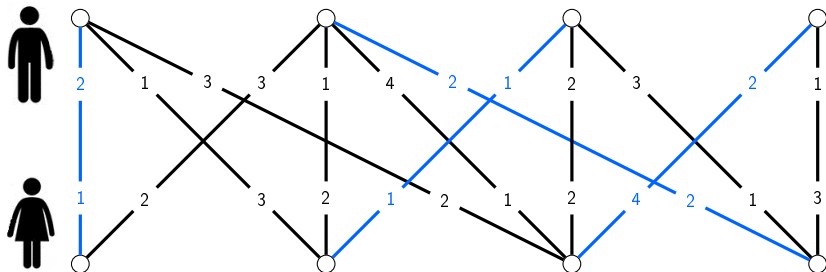
Definition

Edge mw is **blocking** if

- ① it is not in the matching and
- ② m prefers w to his wife or he is single and
- ③ w prefers m to her husband or she is single.



The Gale-Shapley algorithm



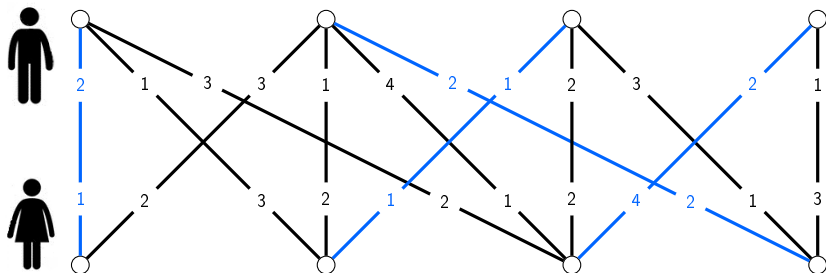
Definition

Edge mw is **blocking** if

- ① it is not in the matching and
- ② m prefers w to his wife or he is single and
- ③ w prefers m to her husband or she is single.



The Gale-Shapley algorithm



Definition

Edge mw is **blocking** if

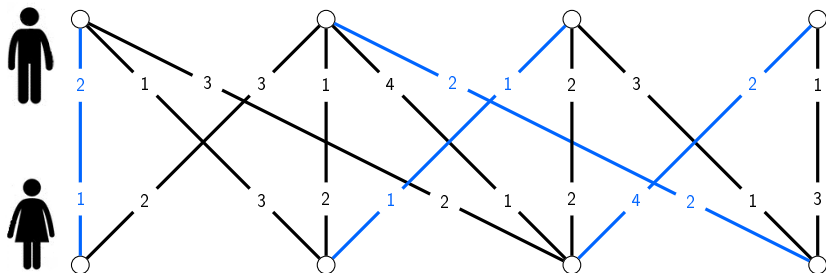
- ① it is not in the matching and
- ② m prefers w to his wife or he is single and
- ③ w prefers m to her husband or she is single.



Extensions:

- non-bipartite instances

The Gale-Shapley algorithm



Definition

Edge mw is **blocking** if

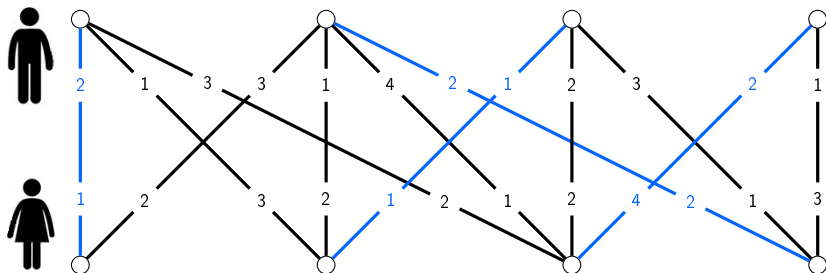
- ① it is not in the matching and
- ② m prefers w to his wife or he is single and
- ③ w prefers m to her husband or she is single.



Extensions:

- non-bipartite instances → stable roommates problem

The Gale-Shapley algorithm



Definition

Edge mw is **blocking** if

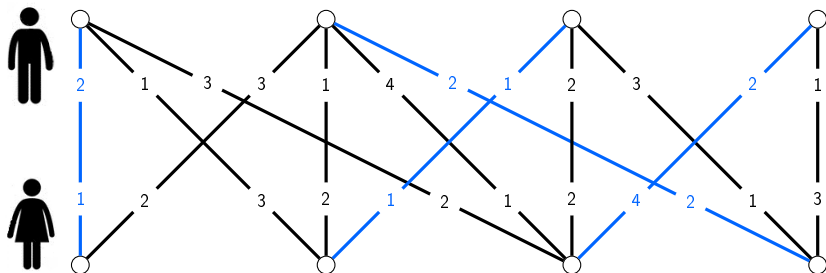
- ① it is not in the matching and
- ② m prefers w to his wife or he is single and
- ③ w prefers m to her husband or she is single.



Extensions:

- non-bipartite instances → stable roommates problem
- edge weights

The Gale-Shapley algorithm



Definition

Edge mw is **blocking** if

- ① it is not in the matching and
- ② m prefers w to his wife or he is single and
- ③ w prefers m to her husband or she is single.



Extensions:

- non-bipartite instances → stable roommates problem
- edge weights → weighted stable matching problem

Theorem (Rural hospitals theorem, Gale, Sotomayor 1985)

The set of matched agents is the same in all stable matchings, even in non-bipartite instances.

Theorem (Rural hospitals theorem, Gale, Sotomayor 1985)

The set of matched agents is the same in all stable matchings, even in non-bipartite instances.

Weighted stable matching

Theorem (Rural hospitals theorem, Gale, Sotomayor 1985)

The set of matched agents is the same in all stable matchings, even in non-bipartite instances.

Weighted stable matching

Theorem (Feder 1992)

In non-bipartite instances, finding a stable matching with maximum weight (among all stable matchings) is NP-hard.

Theorem (Rural hospitals theorem, Gale, Sotomayor 1985)

The set of matched agents is the same in all stable matchings, even in non-bipartite instances.

Weighted stable matching

Theorem (Feder 1992)

In non-bipartite instances, finding a stable matching with maximum weight (among all stable matchings) is NP-hard.

Theorem (Irving, Feder 1994)

In bipartite instances, a stable matching with maximum weight (among all stable matchings) can be found in polytime.



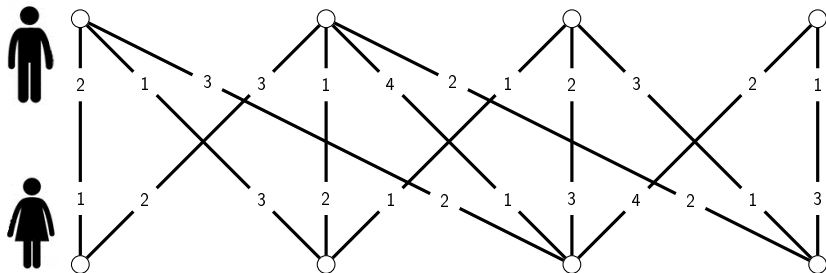
National Resident Matching Program



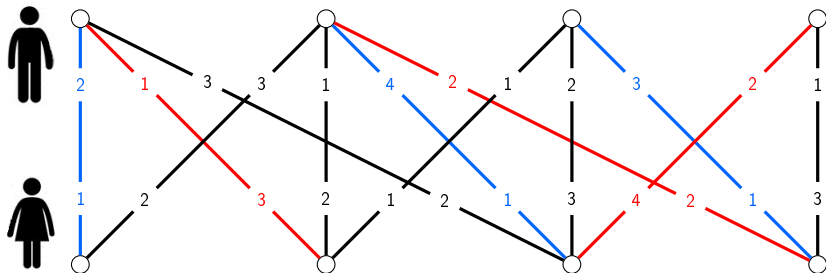
National Resident Matching Program

- non-profit organization created in 1952 in the U.S.
- goal: match medical school graduates to residency positions
- over 41000 students in 2015
- many apply in couples
- need to negotiate stability and size

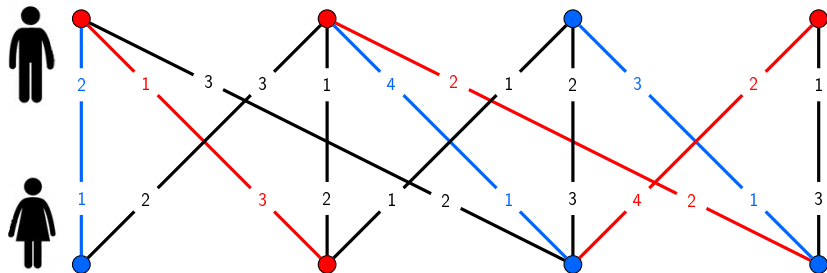
Definition



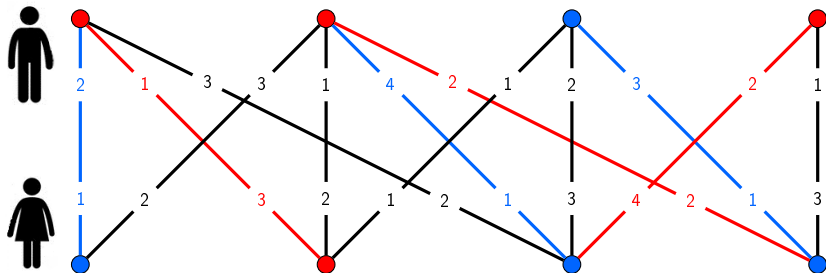
Definition



Definition

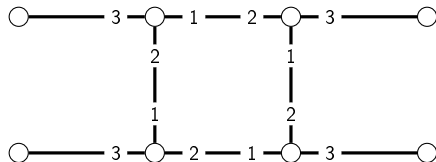


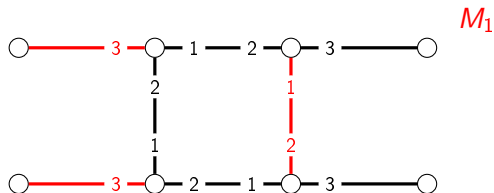
Definition

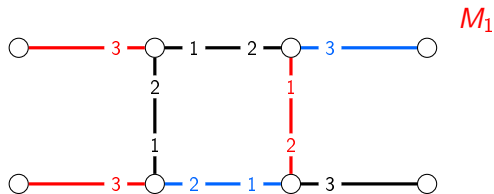


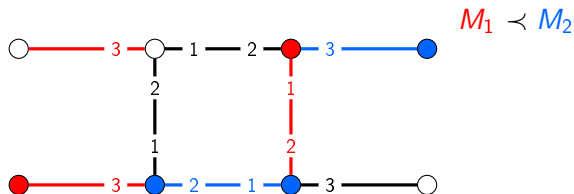
Definition

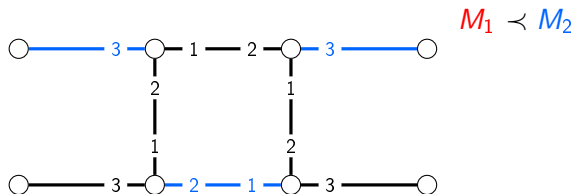
M is popular, if it is at least as popular as any other matching.

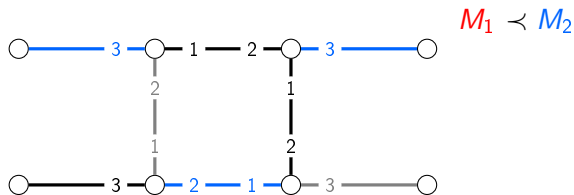


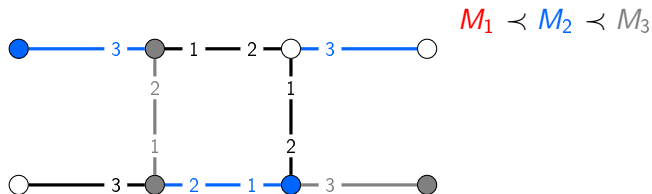


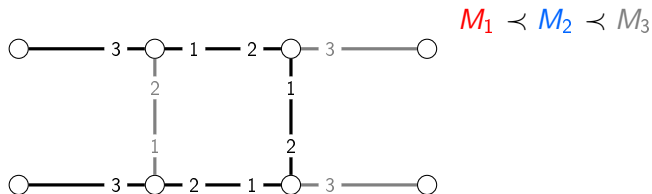


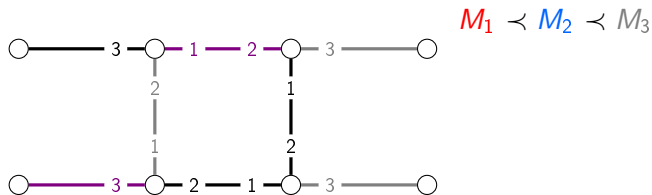


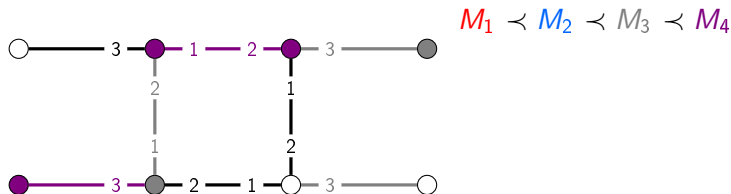


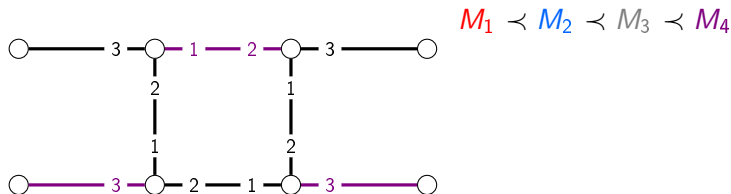


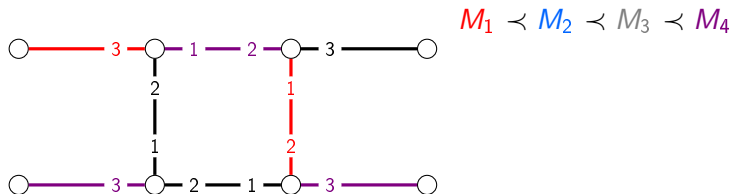


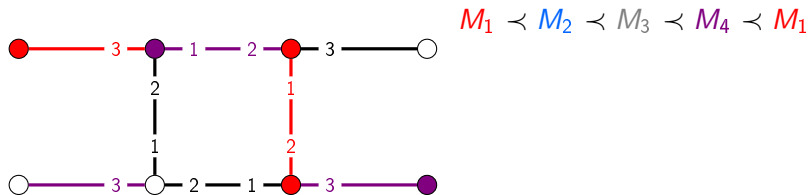


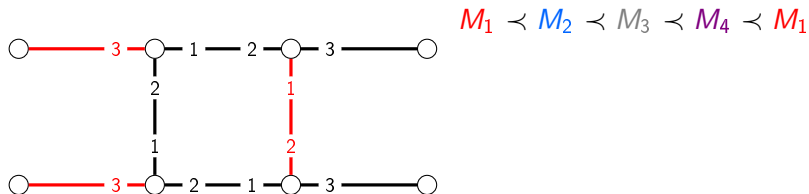


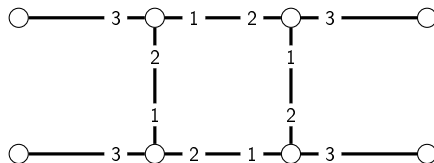








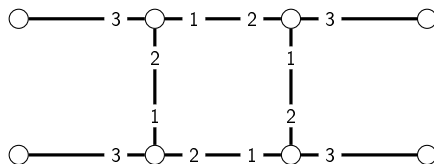




$$M_1 \prec M_2 \prec M_3 \prec M_4 \prec M_1$$

Popular matchings:

- 2 stable matchings of size 2
- the perfect matching of size 4



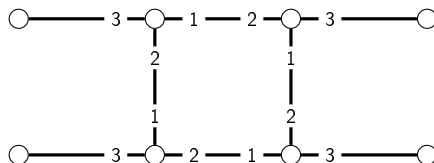
$$M_1 \prec M_2 \prec M_3 \prec M_4 \prec M_1$$

Popular matchings:

- 2 stable matchings of size 2
- the perfect matching of size 4

Theorem (Gärdenfors 1975)

A popular matching always exists.



$$M_1 \prec M_2 \prec M_3 \prec M_4 \prec M_1$$

Popular matchings:

- 2 stable matchings of size 2
- the perfect matching of size 4

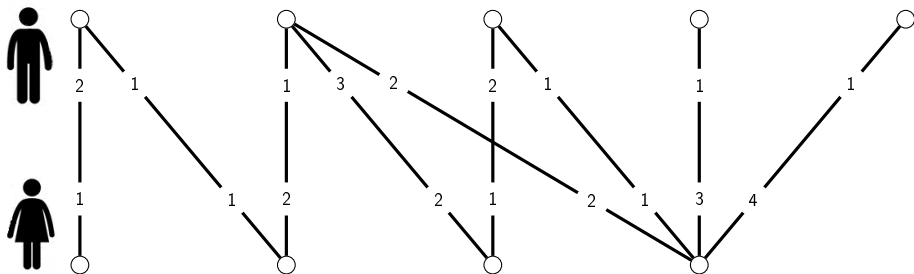
Theorem (Gärdenfors 1975)

A popular matching always exists.

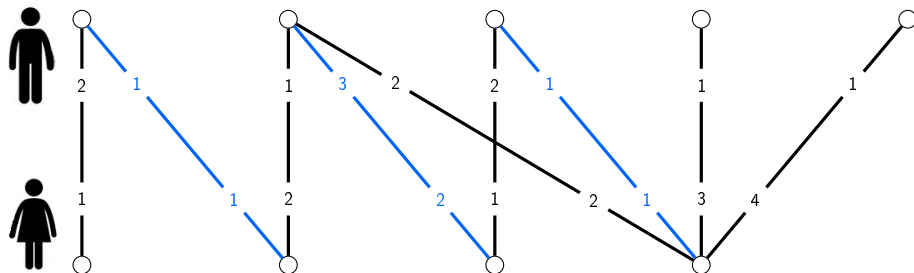
Theorem (Biró, Irving, Manlove 2010)

Stable matchings are minimum size popular matchings.

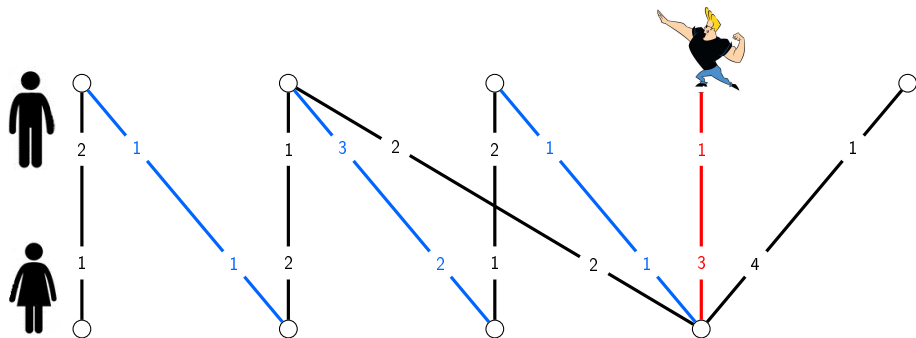
Maximum size popular matching



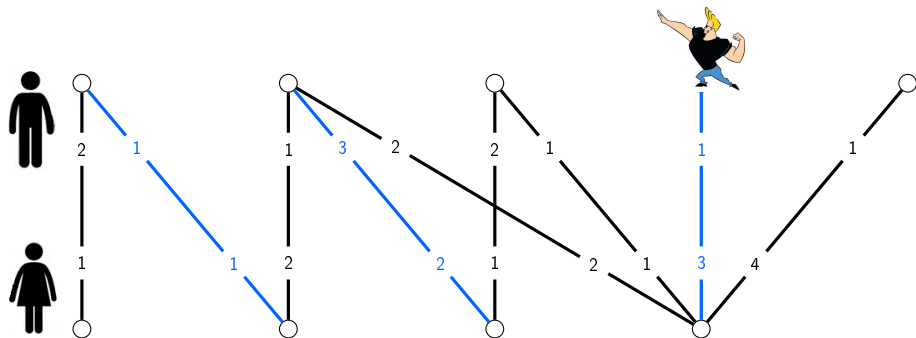
Maximum size popular matching



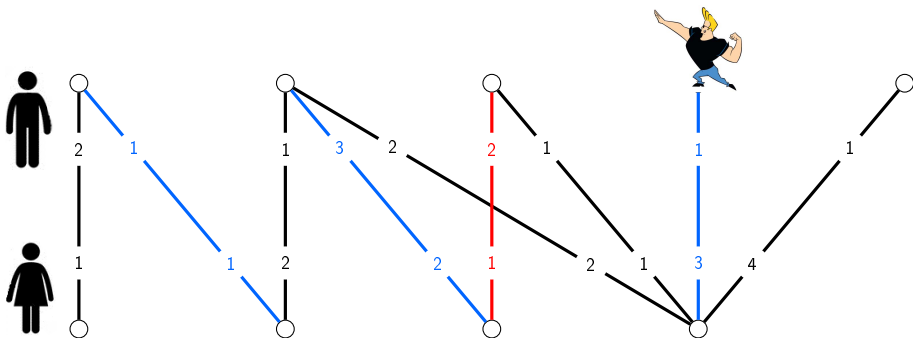
Maximum size popular matching



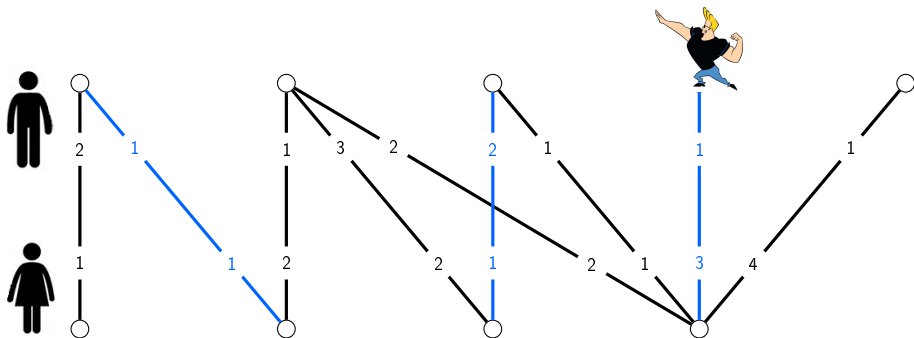
Maximum size popular matching



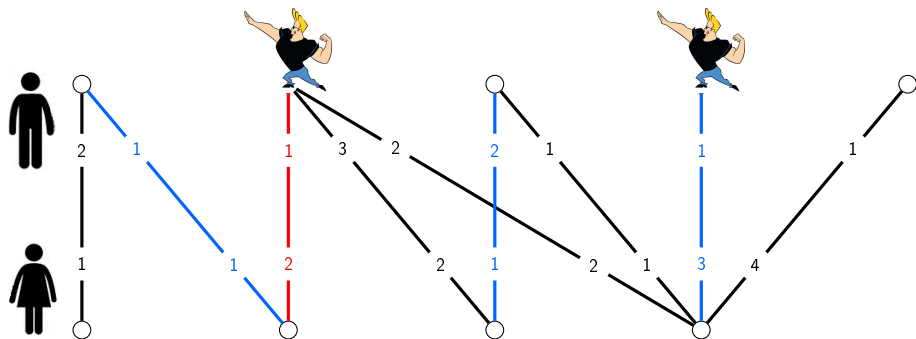
Maximum size popular matching



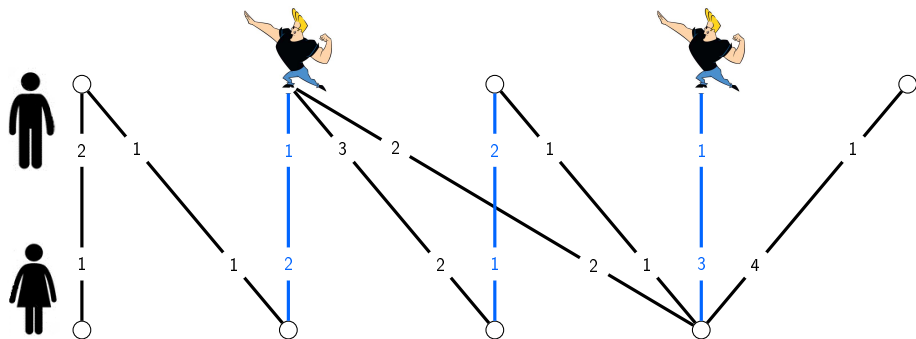
Maximum size popular matching



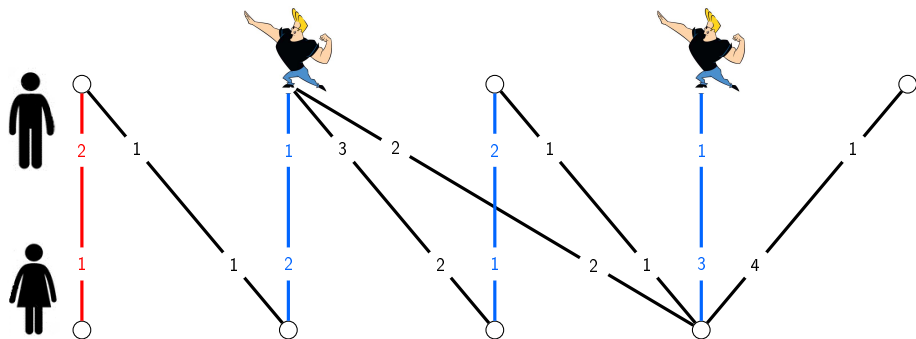
Maximum size popular matching



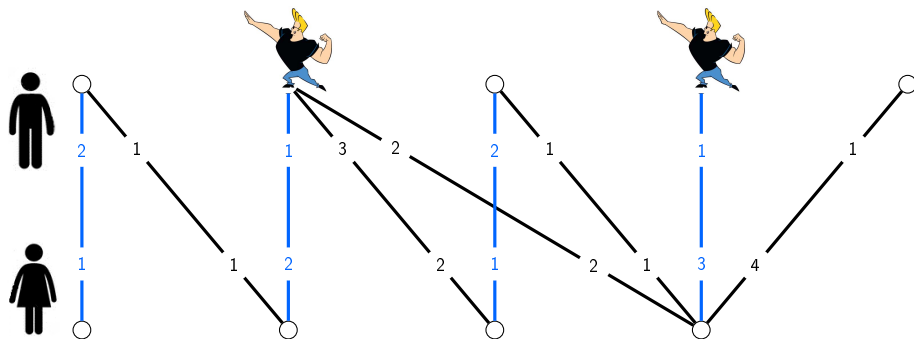
Maximum size popular matching



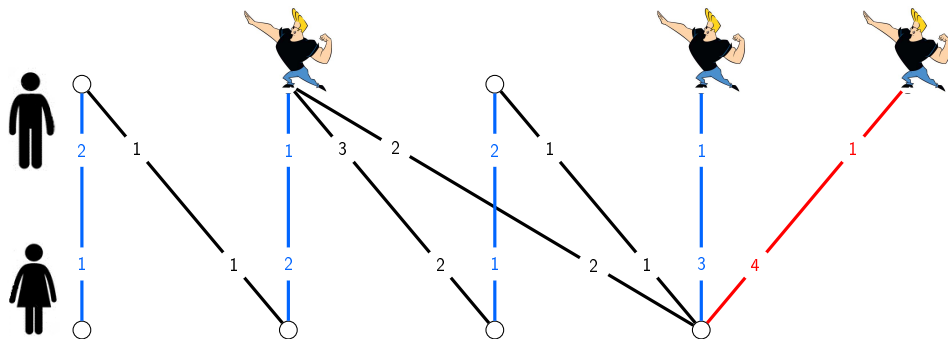
Maximum size popular matching



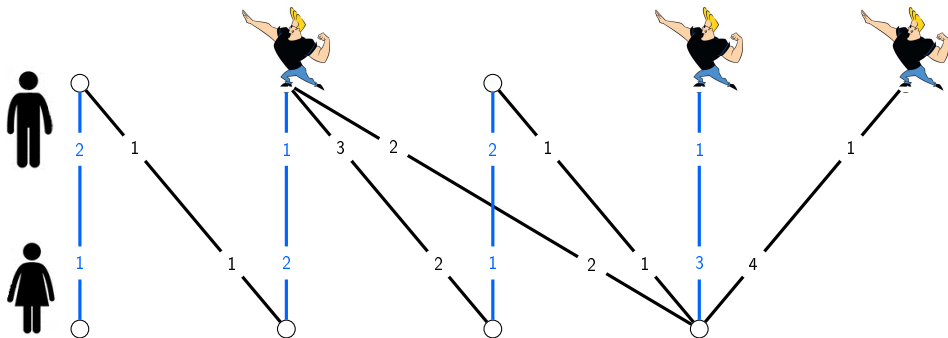
Maximum size popular matching



Maximum size popular matching



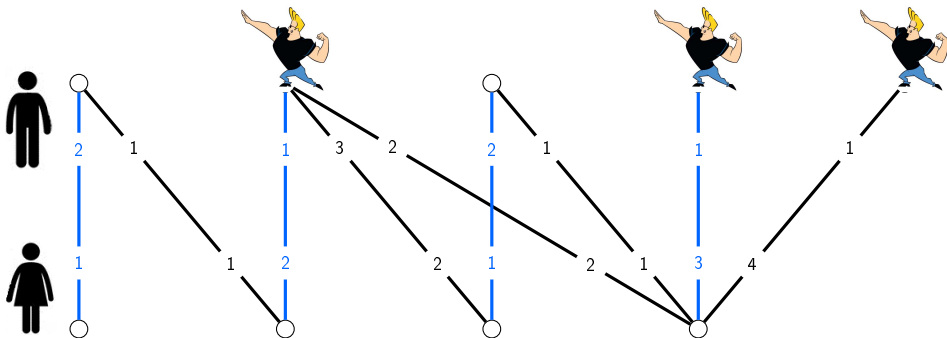
Maximum size popular matching



Theorem (Kavitha 2012)

A max size popular matching can be computed in linear time.

Maximum size popular matching



Theorem (Kavitha 2012)

A max size popular matching can be computed in linear time.

Theorem (Kavitha 2012)

More Gale-Shapley runs \rightarrow larger, less popular matching.

Theorem (Brandl&Kavitha, Nasre&Rawat 2016+)

The same algorithm works for many-to-one matchings.

Theorem (Brandl&Kavitha, Nasre&Rawat 2016+)

The same algorithm works for many-to-one matchings.

Future direction

Extension to many-to-many matchings, stable allocations and stable flows?

Theorem (Brandl&Kavitha, Nasre&Rawat 2016+)

The same algorithm works for many-to-one matchings.

Future direction

Extension to many-to-many matchings, stable allocations and stable flows?

Theorem (Hirakawa, Yamauchi, Kijima, Yamashita 2015)

The same vertices are matched in all max size popular matchings.

Theorem (Brandl&Kavitha, Nasre&Rawat 2016+)

The same algorithm works for many-to-one matchings.

Future direction

Extension to many-to-many matchings, stable allocations and stable flows?

Theorem (Hirakawa, Yamauchi, Kijima, Yamashita 2015)

The same vertices are matched in all max size popular matchings.

Future direction

Given a vertex set S , is there a popular matching that covers exactly S ?

Theorem (Kavitha 2016)

Polytime algorithm for min-cost popular half-integral matching.

Theorem (Kavitha 2016)

Polytime algorithm for min-cost popular half-integral matching.

Future direction

How to optimize over popular matchings when edge costs are present? Is there an LP?

Theorem (Kavitha 2016)

Polytime algorithm for min-cost popular half-integral matching.

Future direction

How to optimize over popular matchings when edge costs are present? Is there an LP?

Theorem (Biró, Irving, Manlove 2010; Cs., Huang, Kavitha 2015)

When preference lists admit ties, the problem of determining whether the instance admits a popular matching is NP-complete. If one side has full ties only, the problem is solvable in polytime.

Theorem (Kavitha 2016)

Polytime algorithm for min-cost popular half-integral matching.

Future direction

How to optimize over popular matchings when edge costs are present? Is there an LP?

Theorem (Biró, Irving, Manlove 2010; Cs., Huang, Kavitha 2015)

When preference lists admit ties, the problem of determining whether the instance admits a popular matching is NP-complete. If one side has full ties only, the problem is solvable in polytime.

Future direction

Where is the boundary between solvable and hard cases?

Slides skipped due to time constraints.

Question (forced edge)

Given an edge e , is there a popular matching M such that $e \in M$?

Slides skipped due to time constraints.

Question (forced edge)

Given an edge e , is there a popular matching M such that $e \in M$?

Question (forbidden edge)

Given an edge e , is there a popular matching M such that $e \notin M$?

Slides skipped due to time constraints.

Question (forced edge)

Given an edge e , is there a popular matching M such that $e \in M$?

Question (forbidden edge)

Given an edge e , is there a popular matching M such that $e \notin M$?

Theorem (Cs., Kavitha 2016)

There is a popular matching M such that $e \in M \Leftrightarrow$

Slides skipped due to time constraints.

Question (forced edge)

Given an edge e , is there a popular matching M such that $e \in M$?

Question (forbidden edge)

Given an edge e , is there a popular matching M such that $e \notin M$?

Theorem (Cs., Kavitha 2016)

There is a popular matching M such that $e \in M \Leftrightarrow$

- there is a stable matching M_1 such that $e \in M_1$ or*

Slides skipped due to time constraints.

Question (forced edge)

Given an edge e , is there a popular matching M such that $e \in M$?

Question (forbidden edge)

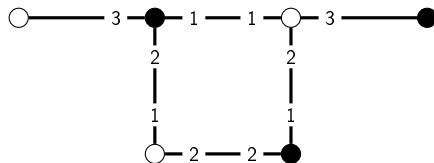
Given an edge e , is there a popular matching M such that $e \notin M$?

Theorem (Cs., Kavitha 2016)

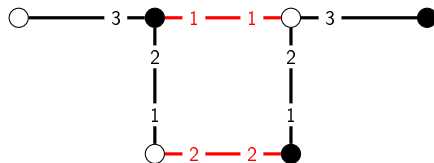
There is a popular matching M such that $e \in M \Leftrightarrow$

- there is a stable matching M_1 such that $e \in M_1$ or*
- there is a **dominant** matching M_2 such that $e \in M_2$.*

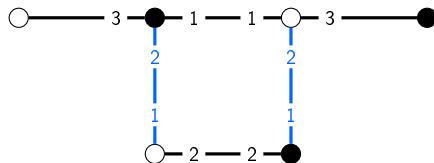
Slides skipped due to time constraints.



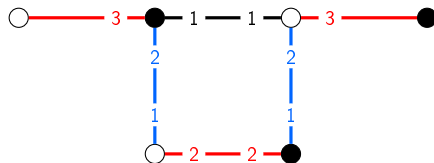
Slides skipped due to time constraints.



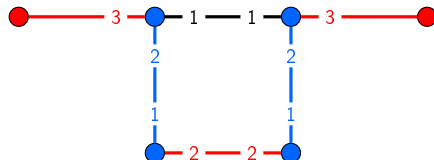
Slides skipped due to time constraints.



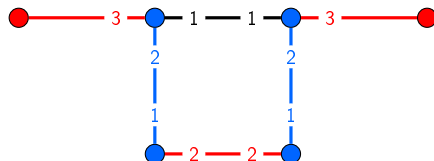
Slides skipped due to time constraints.



Slides skipped due to time constraints.



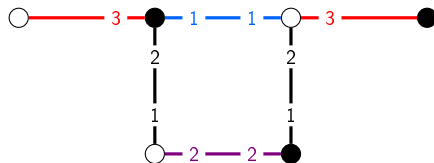
Slides skipped due to time constraints.



- $\begin{array}{|c|} \hline \\ \hline \end{array} \begin{array}{|c|} \hline \\ \hline \end{array}$ is smaller, but strictly more popular than $\begin{array}{c} \text{---} \\ \text{---} \end{array}$

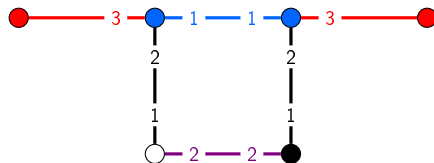
Dominant matchings

Slides skipped due to time constraints.



• $\begin{array}{|c|c|} \hline & \\ \hline \end{array}$ is smaller, but strictly more popular than $\begin{array}{cc} & \\ \hline \end{array}$

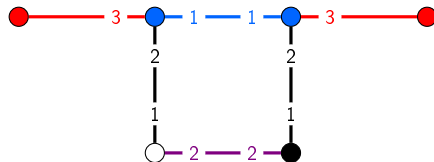
Slides skipped due to time constraints.



- $\begin{array}{|c|c|} \hline \\ \hline \end{array}$ is smaller, but strictly more popular than $\begin{array}{c} \\ \hline \end{array}$

Dominant matchings

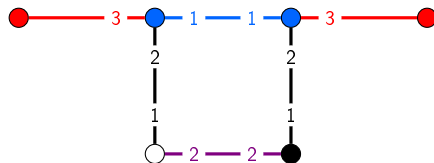
Slides skipped due to time constraints.



- $||$ is smaller, but strictly more popular than $_{-}$
- $_{-}$ is smaller, and not less popular than $_{-}$

Dominant matchings

Slides skipped due to time constraints.

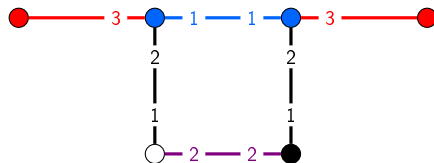


- $||$ is smaller, but strictly more popular than $_{-}$
- $_{-}$ is smaller, and not less popular than $_{-}$

Definition

M *dominates* M' if

Slides skipped due to time constraints.



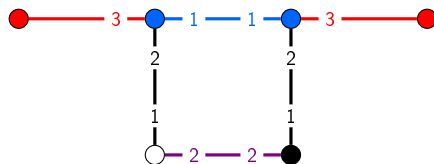
- $| |$ is smaller, but strictly more popular than $-_-$
- $-_-$ is smaller, and not less popular than $-_-$

Definition

M *dominates* M' if

- 1 M is strictly more popular than M' or

Slides skipped due to time constraints.



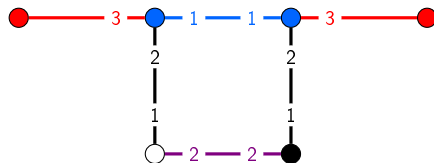
- $|$ $|$ is smaller, but strictly more popular than $_ _$
- $_ _$ is smaller, and not less popular than $_ _$

Definition

M *dominates* M' if

- 1 M is strictly more popular than M' or
- 2 M and M' are equally popular and $|M| > |M'|$.

Slides skipped due to time constraints.



- $| |$ is smaller, but strictly more popular than $-_-$
- $-_-$ is smaller, and not less popular than $-_-$

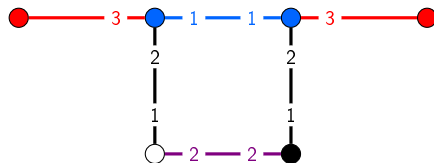
Definition

M *dominates* M' if

- 1 M is strictly more popular than M' or
- 2 M and M' are equally popular and $|M| > |M'|$.

A matching is *dominant* if no other matching dominates it.

Slides skipped due to time constraints.



- $||$ is smaller, but strictly more popular than $-_$
- $_$ is smaller, and not less popular than $-_$

Definition

M *dominates* M' if

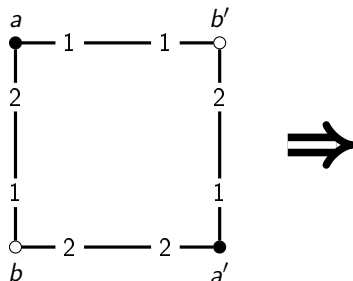
- 1 M is strictly more popular than M' or
- 2 M and M' are equally popular and $|M| > |M'|$.

A matching is *dominant* if no other matching dominates it.

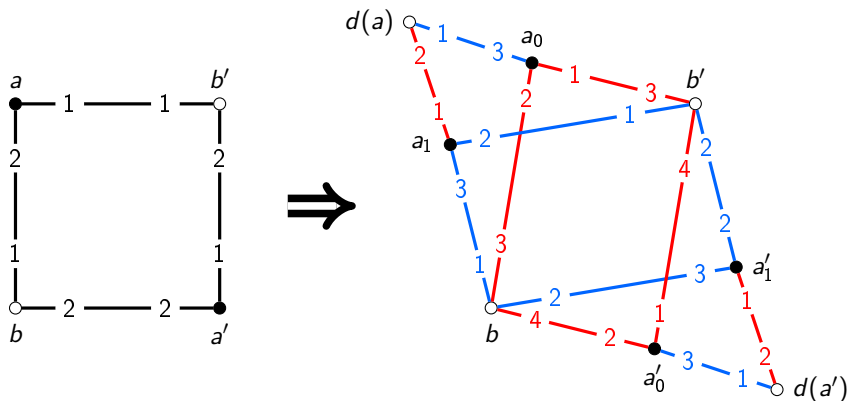
Theorem (Cs., Kavitha 2016)

Dominant matchings exist in every instance.

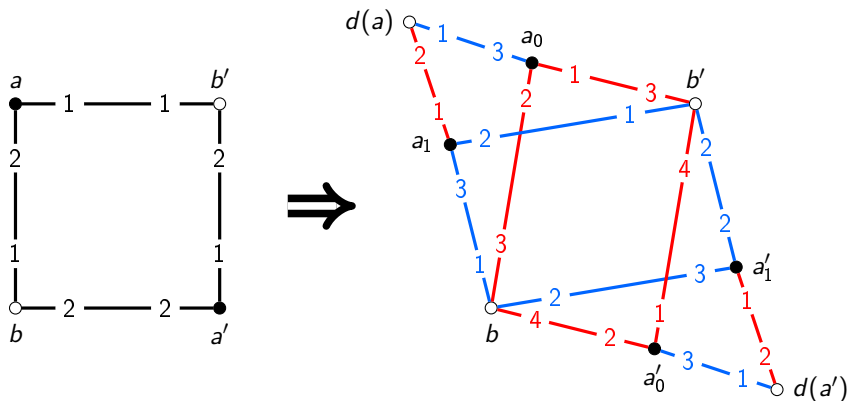
Slides skipped due to time constraints.



Slides skipped due to time constraints.



Slides skipped due to time constraints.



dominant matching \leftrightarrow stable matching

Slides skipped due to time constraints.

What are dominant matchings good for?

Slides skipped due to time constraints.

What are dominant matchings good for?

- Given a forced/forbidden edge ab in G ,
is there a popular matching containing/avoiding ab ?

Slides skipped due to time constraints.

What are dominant matchings good for?

- Given a forced/forbidden edge ab in G ,
is there a popular matching containing/avoiding ab ?
- Lattice structure on stable matchings \rightarrow optimization over the set of dominant matchings (edge weights).

Slides skipped due to time constraints.

What are dominant matchings good for?

- Given a forced/forbidden edge ab in G ,
is there a popular matching containing/avoiding ab ?
- Lattice structure on stable matchings \rightarrow optimization over the set of dominant matchings (edge weights).
- Given G , is there an unstable popular matching?

Slides skipped due to time constraints.

What are dominant matchings good for?

- Given a forced/forbidden edge ab in G ,
is there a popular matching containing/avoiding ab ?
- Lattice structure on stable matchings \rightarrow optimization over the set of dominant matchings (edge weights).
- Given G , is there an unstable popular matching?
If yes, there is an unstable dominant matching.

Open problems

Open problems

- 1 Is there a popular many-to-many matching, stable allocation or stable flow?

Open problems

- 1 Is there a popular many-to-many matching, stable allocation or stable flow?
- 2 Given a vertex set S , is there a popular matching that covers exactly S ? Is there a popular matching of size exactly t ?

Open problems

- 1 Is there a popular many-to-many matching, stable allocation or stable flow?
- 2 Given a vertex set S , is there a popular matching that covers exactly S ? Is there a popular matching of size exactly t ?
- 3 How to optimize over popular matchings when edge costs are present? Is there an LP?

Open problems

- 1 Is there a popular many-to-many matching, stable allocation or stable flow?
- 2 Given a vertex set S , is there a popular matching that covers exactly S ? Is there a popular matching of size exactly t ?
- 3 How to optimize over popular matchings when edge costs are present? Is there an LP?
- 4 When ties are present, where is the boundary between solvable and hard cases?

Open problems

- 1 Is there a popular many-to-many matching, stable allocation or stable flow?
- 2 Given a vertex set S , is there a popular matching that covers exactly S ? Is there a popular matching of size exactly t ?
- 3 How to optimize over popular matchings when edge costs are present? Is there an LP?
- 4 When ties are present, where is the boundary between solvable and hard cases?
- 5 Is there a popular matching containing 2 forced edges?

Open problems

- 1 Is there a popular many-to-many matching, stable allocation or stable flow?
- 2 Given a vertex set S , is there a popular matching that covers exactly S ? Is there a popular matching of size exactly t ?
- 3 How to optimize over popular matchings when edge costs are present? Is there an LP?
- 4 When ties are present, where is the boundary between solvable and hard cases?
- 5 Is there a popular matching containing 2 forced edges?
- 6 Is there a popular matching in the non-bipartite case?