Coherent Measures of Risk from a General Equilibrium Perspective[*]

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Abstract

Coherent measures of risk defined by the axioms of monotonicity, subadditivity, positive homogeneity, and translation invariance are recent tools in risk management to assess the amount of risk agents are exposed to. If they also satisfy law invariance and comonotonic additivity, then we get a subclass of them: spectral measures of risk. Expected shortfall is a well-known spectral measure of risk.

We investigate the above mentioned six axioms using tools from general equilibrium (GE) theory. Coherent and spectral measures of risk are compared to the natural measure of risk derived from an exchange economy model, that we call GE measure of risk. We prove that GE measures of risk are coherent measures of risk. We also show that spectral measures of risk are GE measures of risk only under stringent conditions, since spectral measures of risk do not take the regulated entity's relation to the market portfolio into account. To give more insights, we characterize the set of GE measures of risk via the pricing kernel property.

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1 Introduction

Risk management is of crucial importance considering the enormous financial risk our economy is exposed to. The risks of many economic agents are regulated by various institutions.

For example, if a financial trader wants to sell options, which give the buyer rights of buying or selling at a given price during a specified time horizon (or at a given time), he has to fulfil margin requirements, i.e. he has to deposit some cash or some other riskless and liquid instrument. An exchange's clearing firm, which is responsible for the promises to all parties of transactions being securely completed, requires margin deposits. A measure of risk can be used to determine the margin requirement. The riskier the trader's portfolio, the more the margin requirement should be.

Other external regulators, at an international level, are the International Actuarial Association (IAA) and the International Accounting Standards Board (IASB), who determine the capital requirements for insurance companies. Similarly, the Basel Committee gives guidelines for the acceptable level of capital on banking supervision. Since a government or central bank could be a lender of last resort for these institutions, and the default of them could cause serious problems, they are regulated as well. As an internal regulator, a portfolio manager has to regulate the risk of its traders. In the context of a multi-division firm setting, the head-office may also set risk-limits for the divisions. Internally the risk values can also be used for planning and performance evaluation. It is therefore crucial to measure risk in an appropriate way.

We will use the term *portfolio* when referring to a risky entity (portfolio, firm, insurance company, bank, etc.). The value of a portfolio might change due to all kinds of uncertain events. We relate *risk* to the probability distribution of the future value of the portfolio. For the sake of simplicity in this paper we use discrete random variables. A *measure of risk* assigns a real number to a random variable. It is the minimal amount of cash the regulated agent has to add to his portfolio, and to invest in a zero coupon bond. *Coherent measures of risk* (Artzner, Delbaen, Eber, and Heath, 1999) are defined by four axioms: monotonicity, subadditivity, positive homogeneity and translation invariance. Adding two more axioms: law invariance and comonotonic additivity we get a subclass of coherent measures of risk, namely *spectral measures of risk* (Acerbi, 2002). Expected shortfall is a well-known spectral measure of risk (Acerbi and Tasche, 2002).

Our approach is to model the situation at hand as an exchange economy in a general equilibrium (GE) setting, and determine which axioms are compatible with this model, and whether other axioms emerge as natural. This approach has the advantage that it recognizes the fact that the risk of a portfolio depends on the other assets present in the economy (the *market portfolio*), an insight that is generated immediately by the Capital Asset Pricing Model as developed by Sharpe (1964) and Lintner (1965). By doing so we would like to contribute to the research agenda that connects finance to GE theory, see for instance Geanakoplos and Shubik (1990), Magill and Quinzii (1996), or Leroy and Werner (2001). The corresponding measure of risk of a portfolio would be the amount of cash needed to sell the risk involved in the portfolio to the market. More precisely, the so-called *GE measure of risk* of a portfolio would be the negative of its equilibrium market price.

We prove that GE measures of risk are coherent and comonotonic additive measures of risk. However, GE measures of risk fail to satisfy law invariance, i.e. they are functions of not only the probability distributions of the portfolios, since they also take the regulated entity's relation to the market portfolio into account. Nevertheless we show that GE measures of risk satisfy a generalized notion of law invariance. To check on which domain spectral measures of risk are GE measures of risk, we consider a general domain for the measures of risk. We find that the corresponding domain is very small. To give more insights, we characterize GE measures of risk as the only measures of risk satisfying the property that we call the pricing kernel property.

The structure of the paper is as follows. In Section 2 we discuss coherent measures of risk. In Section 3 spectral measures of risk are discussed. Using the exchange economy model of Section 4 the properties of GE measures of risk are investigated in Section 5. In Section 6 we show that spectral measures of risk are GE measures of risk only under stringent conditions, and we characterize GE measures of risk via the pricing kernel property. We conclude in Section 7.

2 Coherent Measures of Risk

Consider a set $V \subseteq \mathbb{R}^S$ of realization vectors, where S denotes the number of states of nature. State of nature s occurs with probability $p_s > 0$ and $\sum_{s=1}^{S} p_s = 1$. The vector $X \in V$ represents a portfolio's (firm's, insurance company's, bank's, etc.) possible profit and loss realizations on a common chosen future time horizon, say at $t = 1$. The amount X_s if positive (negative) is the portfolio's profit (loss) in state of nature s. The inequality $Y \geq X$ means that $Y_s \geq X_s$ for all $s = 1, \ldots, S$. We use the notations R_+, R_{++} and R_- for $R_+ = [0, \infty)$, $R_{++} = (0, \infty)$, $R_- = (-\infty, 0]$, respectively. The discrete random variable generated by $p \in \mathbb{R}_{++}^S$ and $X \in V$ is denoted by $\hat{X}$, i.e. $P(\hat{X} = X_s) = p_s$, for all $s = 1, \ldots, S$.

A *measure of risk* is a function $\rho : V \to \mathbb{R}$ measuring the risk of a portfolio from the perspective of the present ($t = 0$). It is the minimal amount of cash the regulated agent has to add to his portfolio, and to invest in a reference instrument today, such that it ensures that the risk involved in the portfolio is acceptable to the regulator.[1] We assume that the *reference instrument* has payoff 1 in each state of nature at $t = 1$, thus its realization vector is $\mathbf{1} = (1, \ldots, 1) \in V$. The reference instrument is riskless in the "classical sense", having no uncertainty in its payoffs. It is most natural to think of it as a zero coupon bond. The price of the reference instrument, the *discount factor* is denoted by $\delta \in \mathbb{R}_+$.

We adjusted the definition of coherent measures of risk to the discrete case as follows.

Definition 2.1. A function $\rho : V \to \mathbb{R}$ is called a *coherent measure of risk* (Artzner et al., 1999) if it satisfies the following axioms.

1. *Monotonicity*: for all $X, Y \in V$ such that $Y \geq X$, we have $\rho(Y) \leq \rho(X)$.

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[1]The measure of risk can also be negative, meaning that a portfolio remains acceptable if a certain amount of cash is withdrawn from it.

2. *Subadditivity*: for all $X, Y \in V$ such that $X + Y \in V$, we have
   $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

3. *Positive homogeneity*: for all $X \in V, h \in \mathbb{R}_+$ such that $hX \in V$, we have
   $\rho(hX) = h\rho(X)$.

4. *Translation invariance*: for all $X \in V$ and $a \in \mathbb{R}$ such that $X + a\mathbf{1} \in V$, we have
   $\rho(X + a\mathbf{1}) = \rho(X) - \delta a$.

The axioms are motivated as follows.

By monotonicity if a portfolio Y is always worth at least as much as X (event by event), then Y cannot be riskier than X.

Subadditivity says that if we combine two portfolios, the risk is not greater than the sum of the risks associated with each: it captures the notion of diversification. If an exchange's measure of risk would fail to satisfy this property, then a trader could be better off by splitting his position, opening two accounts and decreasing the margin requirements. The same motivation applies to firms, banks, insurance companies, etc. For instance, for the case of internal capital budgeting, if the measure of risk satisfies subadditivity, then the head office can be sure that by setting risk limits to the divisions, the risk of the whole firm will be smaller than the sum of the individual risk limits. This enables the head office to decentralize the risk constraints.

Positive homogeneity requires that portfolio size should linearly influence risk.

Translation invariance ensures that the measure of risk is expressed in appropriate units. It means that investing an amount $\delta a > 0$ (or $\delta a < 0$) of cash into a units of reference instrument at $t = 0$ leads to the payoff vector $a\mathbf{1}$ at $t = 1$ and decreases (respectively: increases) the measure of risk by δa.

For further motivation of the axioms see Artzner et al., 1999.

The four axioms above seem natural, but it is well known that the rather popular Value-at-Risk (VaR) is not subadditive and hence it is not a coherent measure of risk. As a byproduct checking the relevance of the coherency axioms from a general equilibrium perspective enables us to assess this attack on VaR as well.

3   Spectral Measures of Risk

Acerbi (2002) treats *spectral measures of risk* in case of discrete random variables with *equiprobable* outcomes, i.e. when $p_1 = \cdots = p_S = 1/S$, as a special case. He assumes that the discount factor is 1, and the domain is $\mathbb{R}^S$. In this paper we explicitly want to use the discount factor, which for the moment is exogenously determined here as the price of the reference instrument. Since in the general equilibrium model the discount factor will be determined endogenously, we have to multiply Acerbi (2002)'s definition by δ. Our model is a generalization also in the sense that we consider a general domain $V \subseteq \mathbb{R}^S$. The definition of spectral measures of risk with equiprobable outcomes is as follows.

Let us introduce the ordered statistics $X_{s:S}$ given by the ordered values of the S-tuple $X_1, \ldots, X_S$, i.e. $\{X_{1:S}, \ldots, X_{S:S}\} = \{X_1, \ldots, X_S\}$ and $X_{1:S} \leq X_{2:S} \leq \cdots \leq X_{S:S}$.

Definition 3.1. Let the outcomes be equiprobable. Consider a domain $V \subseteq \mathbb{R}^S$, and a vector $\phi \in \mathbb{R}^S$. The measure $M_\phi : V \to \mathbb{R}$ defined by

$$M_\phi(X) = -\delta \sum_{s=1}^{S} \phi_s X_{s:S} \tag{1}$$

is a *spectral measure of risk* if $\phi \in \mathbb{R}^S$ satisfies the conditions

1. Nonnegativity: $\phi_s \geq 0$ for all $s = 1, \ldots, S$,

2. Normalization: $\sum_{s=1}^{S} \phi_s = 1$,

3. Monotonicity : ϕ_s is non-increasing, i.e. $\phi_{s_1} \geq \phi_{s_2}$ if $s_1 < s_2$ and $s_1, s_2 \in \{1, \ldots, S\}$.

Spectral measures of risk are calculated as discounted weighted average losses, with non-increasing weights, with the highest weight on the worst outcome. The weight vector ϕ is the so-called *risk spectrum*. It can be interpreted as expressing the attitude toward risk. As a special case we have the following definition.

Definition 3.2. Let $k \in \{1, \ldots, S\}$. The *k-expected shortfall* of the realization vector X is defined by

$$\mathrm{ES}_k(X) = -\delta \sum_{s=1}^{k} \frac{1}{k} X_{s:S}. \tag{2}$$

The k-expected shortfall is the discounted average of the worst k outcomes. With an unrestricted domain, i.e. if $V = \mathbb{R}^S$, Acerbi (2002) has the following result.

Proposition 3.3. *Assume $V = \mathbb{R}^S$. A measure of risk $M_\phi : V \to \mathbb{R}$ is coherent if and only if it is a spectral measure of risk as defined by Definition 3.1.*

Proof. Acerbi (2002), Theorem 5.3. □

Of course on a restricted domain, i.e. if $V \subset \mathbb{R}^S$, spectral measures of risk still satisfy the four coherency axioms. However, as the following three examples show, depending on V one might find other measures of risk $M_\phi : V \to \mathbb{R}$ that are coherent but not spectral measures of risk.

Example 3.4. As a trivial example, let V contain only one vector, X. In this case all measures of risk, i.e. all functions $\rho : V \to \mathbb{R}$ satisfy the six axioms, since there are no two vectors on which the axioms of monotonicity, subadditivity, etc. could be checked. Note that in this case spectral measures of risk are those functions $\rho : V \to \mathbb{R}$ that satisfy

$$-\frac{\delta}{S} \sum_{s=1}^{S} X_{s:S} \leq \rho(X) \leq -\delta X_{1:S}, \tag{3}$$

since they are at least as large as the discounted average loss of X, and at most as large as the discounted highest loss of X.

Example 3.5. In the proof of Theorem 5.3 (Proposition 3.3 here) Acerbi (2002) shows for the case $V = \mathbb{R}^S$ that if a measure of risk $M_\phi : V \to \mathbb{R}$ is monotone, then its risk spectrum ϕ should be nonnegative. If for a certain s we have that $\phi_s < 0$, then for any $X \in V$ an increase in $X_{s:S}$ gives rise to a higher measure of risk, contradicting monotonicity.

However, for many domains V, positivity is not required for monotonicity. Consider a domain $V = \mathbb{R}_-^{S-1} \times \{0\}$. In this case the best outcome of any portfolio is zero, thus the weight of the best outcome, ϕ_S can be negative, since the best outcome can not be increased. So although ϕ is not nonnegative everywhere, the measure of risk remains monotone.

Example 3.6. IIn the proof of Theorem 5.3 (Proposition 3.3 here) Acerbi (2002) also shows for the case $V = \mathbb{R}^S$ that if a measure of risk $M_\phi : V \to \mathbb{R}$ is translation invariant, then its risk spectrum ϕ should be normalized. If ϕ is not normalized, then it is easy to check that translation invariance fails. It suffices to consider $M_\phi(X)$ and $M_\phi(Y)$, where $Y = X + a\mathbf{1}$, and $a \in \mathbb{R}, a \neq 0$.

For general domains V, normalization is not required for translation invariance. All that is needed is that there are no two vectors X and Y such that $X = Y + a\mathbf{1}$ for some $a \in \mathbb{R}, a \neq 0$. In this case translation invariance does not have bite and normalization can be omitted.

Any measure of risk $M_\phi : V \to \mathbb{R}$ and in particular spectral measures of risk can be shown to satisfy the axioms of *law-invariance* and *comonotonic additivity*, since they use the ordered statistics of a portfolio. The definitions of these two axioms are as follows.

Definition 3.7. A measure of risk $\rho : V \to \mathbb{R}$ is *law invariant* if for all $X, Y \in V$ such that for all $s = 1, \ldots, S$

$$\Pr(\hat{X} = X_s) = \Pr(\hat{Y} = X_s),$$

we have $\rho(X) = \rho(Y)$.

Law invariance of ρ means that the measure of risk is a function of the probability distribution (law) only. Note that two random variables $\hat{X}$ and $\hat{Y}$ can be different despite the fact that they have the same probability distribution. Acerbi (2004) explains that law invariance can be thought of as the property of "being estimable from empirical data". We will point out in Section 5 that the portfolios' relations to the economy as a whole are also important, which will be shown to violate the law invariance axiom. For instance, if two portfolios have different covariances with the market portfolio, then the regulator may perceive their risk differently, even if their probability distribution is the same.

Another characteristic of spectral measures of risk is comonotonic additivity.

Definition 3.8. Two realization vectors $X \in \mathbb{R}^S$ and $Y \in \mathbb{R}^S$ are *comonotonic* if for all $s_1, s_2 \in \{1, \ldots, S\}$

$$(X_{s_1} - X_{s_2})(Y_{s_1} - Y_{s_2}) \geq 0.$$

A map $\rho : V \to \mathbb{R}$ is *comonotonic additive* if given two comonotonic realization vectors $X, Y \in V$ such that $X + Y \in V$ the map displays additivity, i.e. we have $\rho(X + Y) = \rho(X) + \rho(Y)$.

Comonotonicity is a very strong form of dependence that two random variables can display. If two portfolios are comonotonic, then their value will always move up and down together event by event, providing no diversification at all when added to each other. The measure of risk of a portfolio consisting of two comonotonic subportfolios should therefore be equal to the sum of the measures of risk of the subportfolios.

Tasche (2002) shows that in the continuous, non-atomic case spectral measures of risk are the only coherent measures of risk satisfying law-invariance and comonotonic additivity. As his proof requires non-atomic probability distributions, in the discrete, equiprobable case a new proof is required for the analogous statement. Of course the proposition requires unrestricted domain, i.e. $V = \mathbb{R}^S$.

Proposition 3.9. *Assume that the outcomes are equiprobable and $V = \mathbb{R}^S$. Then a measure of risk is coherent, law invariant and comonotonic additive if and only if it is a spectral measure of risk as defined by Definition 3.1.*

Proof.
($\Leftarrow$) The coherency part follows from Proposition 3.3. Law invariance and comonotonic additivity follows from the fact that spectral measures of risk are using the ordered statistics of a realization vector.

($\Rightarrow$) Take any law invariant and comonotonic additive measure of risk $\rho : \mathbb{R}^S \to \mathbb{R}$ satisfying the four coherency axioms. We will show that law invariance and comonotonic additivity of ρ implies that it can be written in the form of Equation (1), i.e. there exists a vector $\phi \in \mathbb{R}^S$ such that for all $X \in \mathbb{R}^S$ we have that

$$\rho(X) = M_\phi(X) = -\delta \sum_{s=1}^{S} \phi_s X_{s:S}. \tag{4}$$

Law invariance of ρ implies that for all $X \in \mathbb{R}^S$ any permutation of X has the same measure of risk. A particular permutation is the ordered statistics vector of X, $[X_{s:S}]$. Using the notation $\bar{X} = \{X \in \mathbb{R}^S | X_1 \leq X_2 \leq \cdots \leq X_S\}$ for the space of ordered statistics we have that for all $X \in \mathbb{R}^S$ the measure of risk ρ can be written as a function $f : \bar{X} \to \mathbb{R}$ of the ordered statistics vector,

$$\rho(X) = f([X_{s:S}]). \tag{5}$$

Comonotonic additivity of ρ implies that f is linear on $\bar{X}$. Choosing S independent vectors in $\bar{X}$ implies that f is additive separable with constant weight functions $\bar{\phi}_1, \ldots, \bar{\phi}_S \in \mathbb{R}$, i.e.

$$f([X_{s:S}]) = \bar{\phi}_1 X_{1:S} + \cdots + \bar{\phi}_S X_{S:S}. \tag{6}$$

Using $\phi_s = -\frac{\bar{\phi}_s}{\delta}, s = 1, \ldots, S$ we have that Equation (4) is satisfied. Proposition 3.3 implies that ρ should be a spectral measure of risk. $\qquad \square$

It is cumbersome to generalize the definition of spectral measures of risk to the case in which the outcomes are *not equiprobable*. Consider the following example.

Example 3.10. In Table 1 we have 4 states of nature with equal probability of occurrence. Portfolios X and Y, their ordered statistics and the risk spectrum ϕ is also given. A straightforward calculation gives $M_\phi(X) = 0$, and $M_\phi(Y) = 3.5\delta$.

s	p_s	X_s	$X_{s:S}$	Y_s	$Y_{s:S}$	ϕ_s
1	0.25	-3	-3	1	-4	0.4
2	0.25	2	2	-4	-4	0.3
3	0.25	2	2	-4	-4	0.2
4	0.25	2	2	-4	1	0.1

Table 1: Spectral measures of risk with *equiprobable* outcomes.

s	p'_s	X'_s	$X'_{s:S}$	Y'_s	$Y'_{s:S}$	ϕ^g_s
1	0.25	-3	-3	1	-4	0.4
2	0.75	2	2	-4	1	0.6

Table 2: Spectral measures of risk with *not equiprobable* outcomes.

Since the portfolios X and Y have the same outcome in states of nature 2, 3 and 4, we can merge those states. This operation leads to Table 2, where the generalized discrete risk spectrum ϕ^g is given by $\phi^g_1 = \phi_1$ and $\phi^g_2 = \phi_2 + \phi_3 + \phi_4$.

By defining $M_{\phi^g}(X) = -\delta \sum_{s=1}^S \phi^g_s X_{s:S}$ we have that $M_\phi(X) = M_{\phi^g}(X') = 0$, but $M_\phi(Y) = 3.5\delta$, whereas $M_{\phi^g}(Y') = \delta$. To avoid this problem, within the discrete setting a different ϕ^g should be specified for all the orderings of the portfolio vectors. This would rather complicate the analysis and would not add much value since with splitting up the states and increasing their number any discrete distribution can be arbitrarily closely approximated with equiprobable states. Thus we will use the discrete, equiprobable version of spectral measures of risk as defined in Definition 3.1 in the sequel.

4 An Exchange Economy Model

Next we discuss the four axioms underlying coherent measures of risk and the additional two axioms of spectral measures of risk from a general equilibrium perspective. We do this by checking the validity of the six axioms on the natural measure of risk derived from an exchange economy model, that we call *General Equilibrium (GE) measure of risk*. The GE measure of risk of a portfolio is the amount of cash needed to sell the risk involved in the portfolio to the market. More precisely, it is the minimal amount of cash needed to add to the portfolio, such that its market price becomes nonnegative. It is easy to see that the GE measure of risk of a portfolio is the negative of its equilibrium market price.

To specify the GE measure of risk we consider an exchange economy model with two time periods ($t = 0, t = 1$) and uncertainty concerning the state of nature in period $t = 1$. We assume that for each state s in the set $\{1, \ldots, S\}$ its probability of occurrence is objectively known, $p_s = 1/S$.[2] Period $t = 0$ is identified with state $s = 0$. There is a unique nondurable commodity (income) in each state of nature $s = 0, \ldots, S$.

The portfolios (firms, insurance companies, banks, etc.) are represented as exogenously given realization vectors in $\mathbb{R}^S$. Their payoff is a profit or loss in state of nature $s = 1, \ldots, S$.

To simplify the analysis, we assume that the economy can be modeled by means of a representative agent. However, it is not difficult to extend the model to the case with heterogenous agents. We assume that the consumption set of the agent is $\mathbb{R}$ in all states of nature. His consumption stream is denoted by $c = (c_0, \ldots, c_S) \in \mathbb{R}^{S+1}$.[3]

The agent's preferences are represented by a von Neumann-Morgenstern utility function $u : \mathbb{R}^{S+1} \to \mathbb{R}$, given by

$$u(c_0, \ldots, c_S) = v_0(c_0) + \bar{\delta} \sum_{s=1}^{S} p_s v(c_s),$$

where $v_0 : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$ are elementary (Bernoulli) utility functions in state 0 and states $s = 1, \ldots, S$ respectively; the scalar $\bar{\delta} \in \mathbb{R}_{++}$ is the subjective discount factor of the agent.

Moreover, the representative agent is characterized by his (aggregate) endowment $\omega = (\omega_0, \ldots, \omega_S) \in \mathbb{R}^{S+1}$. The endowment expresses the agent's consumption possibilities without trade, depending on the realization of the state of nature. To distinguish between the sure zero-period endowment and the random first-period endowment, we define $\widetilde{\omega} = (\omega_1, \ldots, \omega_S)$.

We assume that income transfers across all states are possible, i.e. markets are complete. Without loss of generality we assume that there are S *contingent contracts*.

Definition 4.1. A *contingent contract* for state s $(s = 1, \ldots, S)$ is a promise to deliver one unit of income in state s and nothing otherwise. The price of the contingent contract,

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[2]We shall only employ the equiprobable assumption when comparing GE measures of risk to spectral measures of risk.

[3]Since the unique commodity is income, the consumption vector $c \in \mathbb{R}^{S+1}$ should be interpreted as earnings, and its negative values as losses.

the state price, expressed in units of period 0 income, is denoted by π_s for $s = 1, \ldots, S$. Furthermore, $\pi_0 = 1$.

The agent can sell his endowment ω, from which he can purchase on the contingent markets any consumption stream c satisfying the budget inequality

$$\sum_{s=0}^{S} \pi_s c_s \leq \sum_{s=0}^{S} \pi_s \omega_s. \tag{7}$$

The agent's utility function and his endowment define an *economy* $\mathcal{E} = (u, \omega)$. The equilibrium state prices are determined by the notion of competitive equilibrium.

Definition 4.2. *A competitive equilibrium for an economy* $\mathcal{E} = (u, \omega)$ *is a consumption vector* $c^* = (c_0^*, \ldots, c_S^*)$ *and state price vector* $\pi^* = (\pi_1^*, \ldots, \pi_S^*)$ *that satisfy the following conditions:*

1. $c^* \in \arg\max_c u(c_0, \ldots, c_S)$ *s.t.* $\sum_{s=0}^{S} \pi_s^* c_s \leq \sum_{s=0}^{S} \pi_s^* \omega_s$,

2. $c^* = \omega$.

We present two sufficient conditions for the existence of an equilibrium with nonnegative state prices.

Proposition 4.3. *Consider an economy* $\mathcal{E} = (u, \omega)$. *If the elementary utility function* $v_0 : \mathbb{R} \to \mathbb{R}$ *is strictly monotonic and concave and* $v : \mathbb{R} \to \mathbb{R}$ *is monotone and concave, then a competitive equilibrium* (c^*, π^*) *exists. Moreover,* $\pi_s^* \geq 0$ *for* $s = 1, \ldots, S$.

Proof. Let us denote by $\text{WP}(\omega)$ the bundles weakly preferred to $\omega \in \mathbb{R}^{S+1}$, i.e.

$$\text{WP}(\omega) = \{c \in \mathbb{R}^{S+1} | u(c) \geq u(\omega)\}.$$

Since in equilibrium $c^* = \omega$, we have to show that there exists a state price vector $\pi^* \in \mathbb{R}_+^S$ that weakly separates the endowment ω from $\text{WP}(\omega)$. Since the elementary utility functions v_0 and v are assumed to be concave, the set $\text{WP}(\omega)$ is convex. Since v_0 is strictly monotone, ω is at the boundary of $\text{WP}(\omega)$. Applying the supporting hyperplane theorem we get that there exists a vector $(\pi_0^*, \pi_1^*, \ldots, \pi_S^*) \in \mathbb{R}^{S+1}$ such that for every $c \in \text{WP}(\omega)$ we have that $\sum_{s=0}^{S} \pi_s^* c_s \geq \sum_{s=0}^{S} \pi_s^* \omega_s$. The strict monotonicity of v_0 implies that $\pi_0^* > 0$, and the monotonicity of v implies that $\pi_s^* \geq 0$ for $s = 1, \ldots, S$. Of course $(\pi_0^*, \pi_1^*, \ldots, \pi_S^*)$ can be normalized such that $\pi_0^* = 1$. $\square$

Let $\mathcal{U}$ denote the set of utility functions u in which the elementary utility functions $v_0 : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$ are twice differentiable, $v_0' > 0$, $v_0'' \leq 0$ and $v' \geq 0$, $v'' \leq 0$.

Proposition 4.4. *Consider an economy $\mathcal{E} = (u, \omega)$. If $u \in \mathcal{U}$, then the competitive equilibrium (c^*, π^*) is unique. Moreover,*

$$\pi_s^* = \frac{\frac{\partial u(\omega)}{\partial c_s}}{\frac{\partial u(\omega)}{\partial c_0}} = \frac{\bar{\delta} p_s v'(\omega_s)}{v_0'(\omega_0)} \geq 0, \quad s = 1, \ldots, S.$$

Proof. It follows from elementary microeconomic theory that the first order conditions of the agent's maximization problem are

$$\frac{\frac{\partial u(c^*)}{\partial c_s}}{\frac{\partial u(c^*)}{\partial c_0}} = \frac{\pi_s^*}{\pi_0^*}, \quad s = 1, \ldots, S, \tag{8}$$

that is the marginal rate of substitution equals the corresponding price ratio. In equilibrium $c^* = \omega$, and π_0^* is assumed to be equal to 1. Then Equation (8) can be rewritten as

$$\pi_s^* = \frac{\frac{\partial u(\omega)}{\partial c_s}}{\frac{\partial u(\omega)}{\partial c_0}} = \frac{\bar{\delta} p_s v'(\omega_s)}{v_0'(\omega_0)}, \quad s = 1, \ldots, S. \tag{9}$$

Since $v_0' > 0$, $v' \geq 0$, $\bar{\delta} > 0$, and $p_s > 0$ we get that $\pi_s^* \geq 0$ for $s = 1, \ldots, S$. The second order conditions are satisfied due to the concavity of the elementary utility functions. $\square$

Under the assumptions of Proposition 4.3 or Proposition 4.4 one can define the equilibrium price of any portfolio $Z \in \mathbb{R}^S$.

Definition 4.5. If (c^*, π^*) is a competitive equilibrium of an economy $\mathcal{E} = (u, \omega)$, then the *equilibrium price*, q_Z^* of the portfolio $Z \in \mathbb{R}^S$ is given by $q_Z^* = \sum_{s=1}^{S} \pi_s^* Z_s$.

Note that under the assumption of Proposition 4.4 the equilibrium price of the portfolio $Z \in \mathbb{R}^S$ is *uniquely* given by

$$q_Z^* = \sum_{s=1}^{S} \pi_s^* Z_s = \bar{\delta} \frac{\sum_{s=1}^{S} p_s v'(\omega_s) Z_s}{v_0'(\omega_0)}. \tag{10}$$

As an important special case of Equation (10), the discount factor becomes

$$\delta = q_1^* = \sum_{s=1}^{S} \pi_s^* 1 = \bar{\delta} \frac{\sum_{s=1}^{S} p_s v'(\omega_s)}{v_0'(\omega_0)}. \tag{11}$$

Note that the discount factor is endogenously determined in the exchange economy model.

5   General Equilibrium Measures of Risk

Now we can define the GE measure of risk of a portfolio as the negative of its equilibrium market price.

Definition 5.1. Consider an economy $\mathcal{E} = (u, \omega)$ with competitive equilibrium (c^*, π^*) and a domain V. *The GE measure of risk* is the map $\rho_{u,\omega} : V \to \mathbb{R}$, where $\rho_{u,\omega}(Z) = -q_Z^*$ for any $Z \in V$.

Note that Definition 4.5 and Definition 5.1 imply that for any $Z \in V$

$$\rho_{u,\omega}(Z) = -q_Z^* = -\sum_{s=1}^{S} \pi_s^* Z_s. \tag{12}$$

Notice that $\rho_{u,\omega}$ is a linear function.

Example 5.2. Consider an economy $\mathcal{E} = (u, \omega)$ and a domain V. Let $v_0(c_0) = c_0 - \frac{1}{2}\alpha c_0^2$ and $v(c_s) = c_s - \frac{1}{2}\alpha c_s^2$ for $s = 1, \ldots, S$, where $\alpha \in \mathbb{R}_{++}$ such that $1 - \alpha\omega_s > 0$, $s = 0, \ldots, S$, i.e. the elementary utility functions are quadratic and increasing at $c^* = \omega$. Since $u \in \mathcal{U}$, using Equations (9) and (12) the GE measure of risk of $Z \in V$ is given by

$$\rho_{u,\omega}(Z) = -q_Z^* = -\sum_{s=1}^{S} \pi_s^* Z_s = -\bar{\delta}\frac{\sum_{s=1}^{S} p_s v'(\omega_s) Z_s}{v_0'(\omega_0)} = -\bar{\delta}\frac{\sum_{s=1}^{S} p_s (1 - \alpha\omega_s) Z_s}{(1 - \alpha\omega_0)}. \tag{13}$$

One can also look at the stochastic part of the aggregate endowment $\widetilde{\omega}$ as representing the *market portfolio*, since it captures the aggregate uncertainties. Let us denote the *total return* of any portfolio Z by r_Z, where the total return is the payoff vector divided by the price: $r_Z = Z/q_Z^*$. It is well known (see for instance Geanakoplos and Shubik (1990)) that with a quadratic utility function the CAPM formula holds. It relates the total return of any portfolio to the total return of the market portfolio as follows.

$$E(r_Z) = r_{\mathbf{1}} + \beta_Z(E(r_{\widetilde{\omega}}) - r_{\mathbf{1}}), \text{ where } \beta_Z = \frac{Cov(r_Z, r_{\widetilde{\omega}})}{Var(r_{\widetilde{\omega}})}. \tag{14}$$

Note that the total return of the reference instrument is given by $r_{\mathbf{1}} = 1/\delta$. From $q_Z^* = E(Z)/E(r_Z)$ and Equation (14) it follows that

$$\rho_{u,\omega}(Z) = -q_Z^* = -\frac{E(Z)}{r_{\mathbf{1}} + \beta_Z(E(r_{\widetilde{\omega}}) - r_{\mathbf{1}})}. \tag{15}$$

Thus if the elementary utility functions are quadratic, then the GE measure of risk of portfolio Z is its discounted expected loss $-E(Z)/r_{\mathbf{1}} = -\delta E(Z)$, corrected with its risk relation to the aggregate endowment.

Using the differentiability assumption let us denote the *set of GE measures of risk on V* by G_V, where

$$G_V = \{\rho_{u,\omega} : V \to \mathbb{R} | u \in \mathcal{U}, \omega \in \mathbb{R}^{S+1}\}.$$

Let us discuss some properties of G_V.

Proposition 5.3. *Any $\rho_{u,\omega} \in G_V$ is a coherent measure of risk as defined by Definition 2.1.*

Proof. By Proposition 4.4 $\pi_s^* \geq 0$ for $s = 1, \ldots, S$. Using this fact and the linearity of $\rho_{u,\omega}$ monotonicity, subadditivity and positive homogeneity follows immediately. For translation invariance note that $\delta = \sum_{s=1}^{S} \pi_s^*$ by Equation (11). Then for all $Z \in V$, for all $a \in \mathbb{R}$ such that $Z + a\mathbf{1} \in V$, we have

$$\rho_{u,\omega}(Z + \mathbf{1}a) = -q_{Z+\mathbf{1}a}^* = -\sum_{s=1}^{S} \pi_s^*(Z_s + a) = -\sum_{s=1}^{S} \pi_s^*(Z_s) - \sum_{s=1}^{S} \pi_s^* a =$$
$$= -q_Z^* - \delta a = \rho_{u,\omega}(Z) - \delta a.$$

$\square$

Note that GE measures of risk are coherent measures of risk under the assumptions of Proposition 4.3 as well.

As GE measures of risk are coherent and their linearity also implies comonotonic additivity, so far five out of six axioms of spectral measures of risk are satisfied by them. The sixth one is law invariance. The following example shows that GE measures of risk are not law invariant.

Example 5.4. As a special case of Example 5.2 consider 2 states of nature at $t = 1$ with equal probability of occurrence. Let $\bar{\delta} = 1$ and $\alpha = 1$. Two portfolios, X^1 and X^2, the aggregate endowment, and the state prices calculated by Equation (13) are given in Table 3 below.

s	p_s	X^1	X^2	ω	π_s^*
0	-	-	-	0	1
1	0.5	-1	-2	0.2	0.4
2	0.5	-2	-1	0.5	0.25

Table 3: A GE measure of risk which is not law invariant.

Let us assume that $X^1, X^2 \in V$. It is easy to see that $\hat{X}^1$ and $\hat{X}^2$ have the same probability distribution. However, $\rho_{u,\omega}(X^1) = 0.9 \neq 1.05 = \rho_{u,\omega}(X^2)$, so law invariance fails.

Note that the two portfolios in Table 3 are related differently to the economy. Portfolio X^1 is larger when ω is smaller, whereas X^2 is smaller when ω is smaller. Although the two portfolios have the same mean, X^1 is a better hedge against the aggregate uncertainty. That is why the GE measure of risk of X^1 is smaller than the GE measure of risk of X^2.

One can easily calculate that in Example 5.4 $r_1 = 1.5385$, and $E(r_{\tilde{\omega}}) = 1.7073$. Using those numbers we can apply Equation (15) and calculate the GE measures of risk of X^1 and X^2 through the CAPM formula as it is given in Table 4. Note that β_{X^1} is positive, whereas β_{X^2} is negative.

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	X^1	X^2
$E(X^i)$	-1.5	-1.5
β_{X^i}	0.7593	-0.6508
$\rho_{u,\omega}(X^i)$	0.9	1.05

Table 4: A GE measure of risk and the CAPM formula.

Most of the GE measures of risk are not law invariant. In Example 5.4 all that is needed is that the state prices are different, which is the case in a generic economy. The failure of law invariance tells us that the aggregate endowment (or market portfolio, in terms of CAPM) should be taken into account, when calculating the risk of a portfolio. Nevertheless one can generalize law invariance in such a way that GE measures of risk satisfy them. The point is to look at the joint distribution of a portfolio and the aggregate endowment.

Definition 5.5. A measure of risk $\rho : V \to \mathbb{R}$ is *generalized law invariant* if for all $X, Y \in V$ such that for all $s = 1, \ldots, S$

$$\Pr(\hat{X} = X_s \text{ and } \hat{\omega} = \omega_s) = \Pr(\hat{Y} = X_s \text{ and } \hat{\omega} = \omega_s),$$

we have $\rho(X) = \rho(Y)$.

Proposition 5.6. *GE measures of risk are generalized law invariant.*

Proof. It is easy to see from Equation (10) and Definition 5.1 that any $\rho_{u,\omega} \in G_V$ is generalized law invariant. □

6 Connections: Spectral and GE Measures of Risk

In this section we show the conditions under which spectral measures of risk as defined in Definition 3.1 are GE measures of risk as defined in Definition 5.1. Then we characterize GE measures of risk.

Definition 6.1. *A vector $\omega \in \mathbb{R}^S$ and a set $V \subseteq \mathbb{R}^S$ are comonotonic if ω and Z are comonotonic for every $Z \in V$.*

Proposition 6.2 says that all the spectral measures of risk are GE measures of risk if and only if their domain is comonotonic with the aggregate endowment, i.e. all the vectors in V are comonotonic with ω, their values go up and down together event by event.

Proposition 6.2. *Assume ω is given and it has different outcomes in each state of nature. Moreover, let V contain S linearly independent vectors which are comonotonic with ω. Then for every M_ϕ there exists a GE measure of risk $\rho_{u,\omega} \in G_V$ such that*

$$M_\phi(Z) = \rho_{u,\omega}(Z) \text{ for all } Z \in V$$

if and only if ω and V are comonotonic.

Proof. Since ω has different outcomes in each state of nature by assumption, without loss of generality we can assume that $\omega_1 < \omega_2 < \cdots < \omega_S$.

($\Leftarrow$) We show that comonotonicity is sufficient. Take any spectral measure of risk, M_ϕ and any $Z \in V$. Since ω and V are comonotonic, ω and Z are also comonotonic. Let us search for a GE measure of risk $\rho_{u,\omega} \in G_V$ satisfying the equation $M_\phi(Z) = \rho_{u,\omega}(Z)$. Since ω is given, the only freedom is in choosing $u \in \mathcal{U}$. Using the definitions the requirement is

$$M_\phi(Z) = -\delta \sum_{s=1}^{S} \phi_s Z_{s:S} = -\bar\delta \sum_{s=1}^{S} \frac{p_s v'(\omega_s) Z_s}{v_0'(\omega_0)} = \rho_{u,\omega}(Z). \tag{16}$$

From Equation (11) the discount factor is determined as

$$\delta = \bar\delta \frac{\sum_{s=1}^{S} p_s v'(\omega_s)}{v_0'(\omega_0)}. \tag{17}$$

Using Equation (17) and the assumption that $p_1 = \cdots = p_S = \frac{1}{S}$ Equation (16) reads as

$$-\bar\delta \frac{\sum_{s=1}^{S} \frac{1}{S} v'(\omega_s)}{v_0'(\omega_0)} \sum_{s=1}^{S} \phi_s Z_{s:S} = -\bar\delta \sum_{s=1}^{S} \frac{\frac{1}{S} v'(\omega_s) Z_s}{v_0'(\omega_0)}. \tag{18}$$

After simplifying and rearranging Equation (18) leads to

$$\sum_{s=1}^{S} \phi_s Z_{s:S} = \frac{\sum_{s=1}^{S} v'(\omega_s) Z_s}{\sum_{s=1}^{S} v'(\omega_s)}. \tag{19}$$

In Equation (19) two weighted averages of Z should coincide. Since ω and Z are comonotonic, a larger ω implies a larger or equal Z. Thus the ordered and the non-ordered statistics of Z coincide, i.e. $Z_{s:S} = Z_s$ for all $s = 1, \ldots, S$. The weights on the left-hand side are by definition normalized, nonnegative, and non-increasing. The weights on the right-hand side are also normalized, and it is easy to see that v can be chosen monotone and concave to obtain the same weights.

($\Rightarrow$) We show that comonotonicity is necessary.

By assumption V contains S linearly independent vectors which are comonotonic with ω. Let us denote them by $Z^1, \ldots, Z^{\bar S} \in V$, where $\bar S = S$. Since for all $\bar s = 1, \ldots, \bar S$ we have that $Z^{\bar s}$ and ω and comonotonic, we have that

$$Z_{s:S}^{\bar s} = Z_s^{\bar s} \text{ for all } s = 1, \ldots, S, \tag{20}$$

that is the ordered and the non-ordered statistics of $Z^{\bar s}$ coincide. Since the proposition should hold for any given M_ϕ, let us take a spectral measure of risk with different ϕ_s in each state of nature. We show indirectly that if ω and V are not comonotonic, then there is no GE measure of risk $\rho_{u,\omega} \in G_V$ such that $M_\phi(Z)$ yields the same number as $\rho_{u,\omega}(Z)$ for all $Z \in V$.

If $M_\phi(Z^{\bar{s}})$ yields the same number as $\rho_{u,\omega}(Z^{\bar{s}})$ for $\bar{s} = 1, \ldots, \bar{S}$, then using the logic of Equation (19) and the fact in Equation (20) the following equations are satisfied:

$$\sum_{s=1}^{S} \phi_s Z_s^{\bar{s}} = \frac{\sum_{s=1}^{S} v'(\omega_s) Z_s^{\bar{s}}}{\sum_{s=1}^{S} v'(\omega_s)} \text{ for } \bar{s} = 1, \ldots, \bar{S}. \tag{21}$$

As the equations in (21) are homogeneous, we can assume that $\sum_{s=1}^{S} v'(\omega_s) = 1$. Since the vectors $Z^1, \ldots, Z^{\bar{S}}$ are linearly independent, the equations in (21) determine that

$$v'(\omega_s) = \phi_s \text{ for } s = 1, \ldots, S. \tag{22}$$

Now take any $Z \in V$ which is not comonotonic with ω (such Z exists since ω and V are not comonotonic). It follows from the definition of comonotonicity that there exist states of nature s_1 and s_2 in which

$$\omega_{s_1} < \omega_{s_2} \text{ but } Z_{s_1} > Z_{s_2}. \tag{23}$$

Using Equation (19) and Equation (22) we get that $M_\phi(Z) = \rho_{u,\omega}(Z)$ if and only if

$$\sum_{s=1}^{S} \phi_s Z_{s:S} = \sum_{s=1}^{S} \phi_s Z_s. \tag{24}$$

Both sides of Equation (24) are weighted averages of Z. The left-hand side assigns strictly decreasing weights to the ordered statistics of Z. The right-hand side assigns the same weights but due to the observation in (23) at least one higher outcome of Z gets a higher weight, leading to

$$\sum_{s=1}^{S} \phi_s Z_{s:S} < \sum_{s=1}^{S} \phi_s Z_s. \tag{25}$$

We have a contradiction. □

Note that we had to assume in Proposition 6.2 that ω has different outcomes in each state of nature. In a generic economy this assumption is satisfied. The second assumption was that V contains S linearly independent vectors which are comonotonic with ω. It is easy to check that this assumption is satisfied if V contains all the vectors which are comonotonic with ω or if V contains a convex cone generated by ω and $S - 1$ linearly independent vectors. However, if we only require that V contains ω, then Proposition 6.2 can not be proven as the following example shows.

Example 6.3. Suppose $V = \{\omega, Z\}$, where ω and Z are given in Table 5 below.

Note that the vector ω and V are not comonotonic in Table 5, since $Z \in V$ is not comonotonic with ω. We show that in this example for every M_ϕ there exists a GE measure of risk $\rho_{u,\omega} \in G_V$ such that $M_\phi(Z) = \rho_{u,\omega}(Z)$ for all $Z \in V$, although ω and V

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s	ω_s	Z_s
1	1	-2
2	2	-1
3	3	-1.25

Table 5: Z is not comonotonic with ω.

are not comonotonic. Using the normalized version of Equation (19) we have that M_ϕ and $\rho_{u,\omega}$ lead to the same number for ω if

$$\phi_1 + 2\phi_2 + 3\phi_3 = v'(\omega_1) + 2v'(\omega_2) + 3v'(\omega_3), \text{ and for } Z \text{ if} \tag{26}$$

$$-2\phi_1 - 1.25\phi_2 - \phi_3 = -2v'(\omega_1) - v'(\omega_2) - 1.25v'(\omega_3). \tag{27}$$

Moreover, we know that

$$\phi_1 + \phi_2 + \phi_3 = v'(\omega_1) + v'(\omega_2) + v'(\omega_3) = 1, \text{ and by definition} \tag{28}$$

$$\phi_1 \geq \phi_2 \geq \phi_3. \tag{29}$$

Solving the Equations (26), (27) and (28) leads to the unique solution

$$v'(\omega_1) = \phi_1 + \frac{1}{5}\phi_2 - \frac{1}{5}\phi_3, \tag{30}$$

$$v'(\omega_2) = \frac{3}{5}\phi_2 + \frac{2}{5}\phi_3, \tag{31}$$

$$v'(\omega_3) = \frac{1}{5}\phi_2 + \frac{4}{5}\phi_3. \tag{32}$$

Since v is assumed to be concave,

$$v'(\omega_1) \geq v'(\omega_2) \geq v'(\omega_3) \text{ should hold.} \tag{33}$$

From Equation (29) we have that $\phi_2 \geq \phi_3$. This observation implies with Equation (30) that $v'(\omega_1) \geq \phi_1$; with Equations (31), (32) that $v'(\omega_2) \geq v'(\omega_3)$; and with Equation (31) that $v'(\omega_2) \leq \phi_2$. Since $\phi_1 \geq \phi_2$ from Equation (29) the inequalities in Equation (33) are clearly satisfied.

The strong requirement in Proposition 6.2 is that ω and V should be comonotonic to represent all the spectral measures of risk by GE measures of risk. In this case all the vectors in V are dependent in the strongest form on the aggregate endowment, their value should always go up and down together event by event. The larger the number of states S, the smaller the probability that it occurs.

To give more insights, we characterize the set of GE measures of risk as follows.

Definition 6.4. A measure of risk $\rho : V \to \mathbb{R}$ satisfies the *pricing kernel* property on V if there exists a vector $\gamma \in \mathbb{R}_+^S$ such that for all $Z \in V$ we have that

$$\rho(Z) = -\sum_{s=1}^{S} p_s \gamma_s Z_s. \tag{34}$$

The vector γ is interpreted as the pricing kernel, and the measure of risk of portfolio Z is the negative of its price, induced by the pricing kernel. Let us denote by P_V the set of risk measures satisfying the pricing kernel property on V. Then we have the following proposition.

Proposition 6.5. *The set of GE measures of risk on V coincides with the set of risk measures satisfying the pricing kernel property on V, thus $G_V = P_V$.*

Proof.
$G_V \subseteq P_V$

For any $\rho_{u,\omega} \in G_V$ by Equation (12) we have that

$$\rho_{u,\omega}(Z) = -\sum_{s=1}^{S} \pi_s^* Z_s. \tag{35}$$

By Proposition 4.4 $\pi^* \in \mathbb{R}_+^S$. Using the notation $\gamma_s = \pi_s^*/p_s, s = 1, \ldots, S$ Equation (35) can be rewritten as

$$\rho_{u,\omega}(Z) = -\sum_{s=1}^{S} p_s \gamma_s Z_s, \gamma \in \mathbb{R}_+^S, \tag{36}$$

thus $\rho_{u,\omega}$ satisfies the pricing kernel property.
$G_V \supseteq P_V$

Let us take any measure of risk $\bar{\rho} \in P_V$ satisfying the pricing kernel property on V. By definition there exists a vector $\gamma \in \mathbb{R}_+^S$ such that for all $Z \in V$ we have that

$$\bar{\rho}(Z) = -\sum_{s=1}^{S} p_s \gamma_s Z_s. \tag{37}$$

We show that there exist a quadratic utility function u and endowment $\omega \in \mathbb{R}^S$ such that for all $Z \in V$ we have that $\bar{\rho}(Z) = \rho_{u,\omega}(Z)$. Let $v_0(c_0) = c_0 - \frac{1}{2}\alpha c_0^2$ and $v(c_s) = c_s - \frac{1}{2}\alpha c_s^2$ for $s = 1, \ldots, S$. In equilibrium $c^* = \omega$. We know from Equation (13) that

$$\rho_{u,\omega}(Z) = -\bar{\delta}\frac{\sum_{s=1}^{S} p_s v'(\omega_s) Z_s}{v_0'(\omega_0)} = -\bar{\delta}\frac{\sum_{s=1}^{S} p_s(1 - \alpha\omega_s) Z_s}{(1 - \alpha\omega_0)}. \tag{38}$$

Let $\alpha = 1$, $\bar{\delta} = 1$ and $\omega_0 = 0$. Then Equation (38) simplifies to

$$\rho_{u,\omega}(Z) = -\sum_{s=1}^{S} p_s(1 - \omega_s) Z_s. \tag{39}$$

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By setting ω_s such that $\gamma_s = (1 - \omega_s)$ Equation (37) and Equation (39) define the same measure of risk. As $v'_0 = 1$ and $v'_s = \gamma_s, s = 1, \ldots, S$, the monotonicity requirements of the elementary utility functions are satisfied. $\qquad\square$

Thus GE measures of risk are the only measures of risk satisfying the pricing kernel property. They are linear functions induced by S nonnegative parameters $(\gamma_1, \ldots, \gamma_S)$. On the other hand spectral measures of risk are piecewise linear functions, also induced by S parameters $(\phi_1, \ldots, \phi_S)$, which are perturbed for the different orderings of the portfolio vectors. Thus although the cardinalities of their sets are the same, under some technical assumptions spectral measures of risk are GE measures of risk if and only if the orderings of all the portfolio vectors are the same as the ordering of the market portfolio.

7 Conclusion

In this paper we discussed coherent and spectral measures of risk from a general equilibrium (GE) perspective. Coherent measures of risk can be defined by four axioms: monotonicity, subadditivity, positive homogeneity, translation invariance. Adding two more axioms, law invariance and comonotonic additivity leads to spectral measures of risk. We considered the discrete setting and a general domain $V \subseteq \mathbb{R}^S$. We proved that it is also true in the discrete setting with unrestricted domain, i.e. if $V = \mathbb{R}^S$ that spectral measures of risk are the only coherent measures of risk satisfying law invariance and comonotonic additivity. However, we showed counterexamples that with a general domain V it is not necessarily the case.

We defined a natural measure of risk coming out of a general equilibrium model. The GE measure of risk of a portfolio is the negative of its equilibrium market price. Checking the properties of GE measures of risk enabled us to assess the above mentioned six axioms. We found that GE measures of risk are coherent measures of risk. This way the four axioms of coherent measures of risk are supported from a general equilibrium perspective. Thus the failure of VaR not satisfying subadditivity is strengthened.

However, GE measures of risk do not satisfy law invariance, but only a generalized version of it, in which the market portfolio is also taken into account. Since spectral measures of risk are law invariant, we can conclude that in general, when calculating the risk of a regulated entity spectral measures of risk do not take into account its relation to the market portfolio, leading to an under- or overestimation of risk. The same idea is shown by our result that spectral measures of risk are GE measures of risk if and only if all the regulated entities are comonotonic with the market portfolio, i.e. their value go up and down together event by event. Finally we showed that GE measures of risk are the only measures of risk satisfying the pricing kernel property, which means that any nonnegative pricing kernel can induce them as the negative of the equilibrium market price.

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