

Random Paths to Stability in Hedonic Coalition Formation

Szilvia Pápai*[†]

University of Notre Dame

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Abstract

We study the myopic stability of hedonic coalition formation. Myopic stability means that for each unstable coalition structure there exists a sequence of myopic blockings that leads to a stable coalition structure. The main result is a characterization of the hedonic coalition formation models which are myopically stable whenever a stable coalition structure exists. One interesting implication of this result is that every stable hedonic coalition formation model is myopically stable. Previously known positive results about the marriage model (Roth and Vande Vate, 1990) and the roommate model (Diamantoudi, Miyagawa, and Xue, 2002) also follow.

Keywords: hedonic coalition formation, roommate and marriage problems, stability, core

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[†]Department of Finance, Mendoza College of Business, University of Notre Dame, Notre Dame, IN 46556, USA; e-mail: spapai@nd.edu

1 Introduction

We study the existence of a path to stability in hedonic coalition formation problems. The question is whether, in the absence of a coordinating procedure, a decentralized process of successive myopic blockings leads to a stable (core) coalition structure. More precisely, what we are interested in finding out is whether, starting from an arbitrary coalition structure, using a natural and decentralized process which allows for a randomly selected coalition to form when it is desired by all of its members (a blocking coalition), will we eventually arrive at a coalition structure for which no such blocking coalition exists, and hence the coalition structure is stable? Allowing just one blocking coalition to form at a time is not a limitation of this process, since any blocking coalition that is not chosen to form in some round, and has no common member with the chosen coalition, is still a blocking coalition in the next round, and thus the results can easily be adopted to a process that allows multiple blocking coalitions to form at the same time, whenever the formation of such coalitions is feasible. The process that we are interested in is myopic in the sense that players do not try to foresee how their decision to form a coalition today will affect the decisions of other players tomorrow, or in general, players do not take into account the future evolution of the coalition formation process. This is an assumption on rationality which may well be appropriate in coalition formation processes in which individual players have little knowledge about the preferences of other players.

Roth and Vande Vate (1990) showed that this myopic process yields a stable matching, starting from any initial unstable matching, for marriage problems.¹ Chung (2000) generalized this result by introducing a restriction on preference profiles, called the “no odd rings” condition, and proving that it is sufficient for a roommate problem to be myopically

¹For an introduction to the marriage problem and two-sided matching in general, see Roth and Sotomayor (1990).

stable. Subsequently, Diamantoudi, Miyagawa, and Xue (2002) strengthened both results, although for strict preferences only, by extending them to the roommate model without any additional condition on preferences.² The importance of the Diamantoudi, Miyagawa, and Xue (2002) theorem lies in recognizing that the existence of a myopic path converging to a stable matching is not due exclusively to the two-sided structure of marriage problems, an intuition further supported by Chung’s “no odd rings” condition, but rather, it seems to be a mere implication of the existence of a stable matching, assuming there are no indifferences. As Diamantoudi, Miyagawa, and Xue (2002) point out, their result does not easily extend to hedonic coalition formation, in which coalitions of any size may form. The topic of the current paper is myopic stability, as examined by Roth and Vande Vate (1990), Chung (2000), and Diamantoudi, Miyagawa, and Xue (2002), but in the broader framework of hedonic coalition formation.

The general hedonic coalition formation model (Banerjee, Konishi, and Sönmez, 2001; Bogomolnaia and Jackson, 2002; Cechlárová and Romero-Medina, 2001) is a straightforward generalization of the marriage and roommate models in that it is the same model, except that it allows any coalition to form, regardless of its size or members. In a hedonic coalition formation problem each player ranks the coalitions in which the player is a member. It may appear that a hedonic coalition formation model only takes into account the “hedonic aspect” (Drèze and Greenberg, 1980) of preferences, i.e., the subjective value based on the identity of the members of a coalition. In fact, this simple coalition formation model captures, in essence, many different coalition formation models which explicitly specify the “objective” worth of a coalition, by implicitly taking into account, in the “subjective” preference orderings, all of the “objective” aspects.

Our focus in this paper is on hedonic coalition formation models, as opposed to individual

²Further papers on convergence to the core via a myopic process are Feldman (1974), which studies an exchange economy, and Green (1974), which considers a more abstract environment.

coalition formation problems.³ A hedonic coalition formation model, such as the well-known marriage and roommate models, is given by the feasible coalitions. For example, the roommate model is defined by all coalitions that consist of one or two members only, which are the only feasible coalitions for this model. For the marriage model, a further restriction for a feasible coalition is that each coalition with two members consists of a “man” and a “woman,” where the set of players is divided into “men” and “women.” By contrast, a coalition formation problem is a fixed preference profile, a single instance of (perhaps only privately known) preferences in some coalition formation model. The previous positive results on random paths to stability (what we call myopic stability) are established for specific hedonic coalition formation models (i.e., for the marriage and roommate models), and the question we investigate here is whether these positive results extend to other, more general, hedonic coalition formation models, and if they do, to what extent?

We present one main result, a characterization of all hedonic coalition formation models that ensure myopic stability at each preference profile where a stable coalition structure exists, assuming that preferences are strict. The result is presented and proved with the help of graph representations of hedonic coalition formation. These graph representations are extremely useful both for the formal analysis of this problem and for providing intuition about stability and dynamics in a particular hedonic coalition formation problem. Without graphs, one would have to work with preference profiles which contain much more information than needed for our analysis, and, in the broad framework of general hedonic coalition formation, preference profiles are quite cumbersome and unsatisfactory to work with. This explains the implicit suggestion of Diamantoudi, Miyagawa, and Xue (2002) that extending their analysis to hedonic coalition formation models in general may be difficult. The graphs we use streamline the relevant information about preferences, and provide a simple

³A similar approach was taken in two previous papers (Pápai, 2002; 2003) which explore the existence and uniqueness of stable hedonic coalition structures, respectively.

and intuitive way of identifying stable coalition structures. This graph representation was first introduced in Pápai (2002), and was used there to identify stable coalition formation models. Here we further develop this graph theoretic tool, and represent both hedonic coalition formation problems and models in this concise manner. Moreover, our main characterization theorem is a description of all forbidden subgraphs in the graph representation of hedonic coalition formation models which need to be absent in order to have myopic stability. These forbidden subgraphs represent configurations of coalitions which cannot be feasible simultaneously if myopic stability is desired in the hedonic coalition formation model.

Let us make two clarifying comments about myopic stability. Both our and the previous results establish the existence of a myopic sequence to a stable coalition structure, or, alternatively, that the myopic process converges to a stable coalition structure with probability one, assuming that all blocking coalitions have a positive probability to be selected at each round. Thus, the possibility of cycling through a sequence of coalition structures, prior to reaching a stable coalition structure, is not ruled out by this process (see, for example, Knuth's example presented in Roth and Vande Vate (1990)). The second clarification is about reaching a specific stable coalition structure. Our result, just like the previous ones, does not say that myopic blockings lead to a *specific* stable coalition structure from an arbitrary unstable one, rather, that *some* stable coalition structure is reached.⁴

In addition to gaining insight into the myopic stability of hedonic coalition formation in general, our main theorem also allows us to obtain the two previous positive results for the marriage and roommate models as corollaries. In fact, the main theorem enables us to deduce several interesting secondary results, all of which are sufficient conditions for myopic

⁴It is interesting to note that if we order players and allow blocking coalitions to form according to this order then, as Ma (1996) shows, some stable coalition structures may never be reached via successive myopic blockings, no matter which order of the players we choose.

stability of a coalition formation model for each stable preference profile. One important implication is that any stable coalition formation model (i.e., for which each preference profile has a stable coalition structure) is also myopically stable (Corollary 1). This result, in turn, implies the Roth and Vande Vate (1990) result on marriage models (Corollary 2), but for strict preferences only. Another interesting corollary is a description of three very simple forbidden coalition configurations, in the absence of which we get a positive result (Corollary 3). It is interesting to note that Corollary 1 can also be stated in terms of forbidden coalition configurations, but it prohibits a completely different set of coalition configurations. While Corollary 1 leads to the result on the marriage model, Corollary 3 implies the theorem of Diamantoudi, Miyagawa, and Xue (2002) on the roommate model (Corollary 4), which in turn also yields the result on the marriage model.

2 Hedonic Coalition Formation and Stability

There is a **finite set of players** $N = \{1, \dots, n\}$. Each nonempty subset $S \subseteq N$, $S \neq \emptyset$ is a **coalition**. For all coalitions $S \subseteq N$, let $[S] = \{\{i\} : i \in S\}$ denote the set of singletons for the members of S . Let $\Pi = \{S \subseteq N : S \neq \emptyset, |S| \neq 1\}$ denote the set of all non-singleton coalitions in N . We assume that it is always feasible for a player to stay alone, i.e., players are not forced to join any coalition. Therefore, we exclude the set of singletons from Π for the sake of simplicity. A **collection of feasible** (non-singleton) **coalitions** is given by $\Pi^* \subseteq \Pi$. Each collection of feasible coalitions defines a **hedonic coalition formation model**. For example, the roommate model is given by $\Pi^* = \{S \subseteq N : |S| = 2\}$.

A **coalition structure** $\sigma = \{S_1, \dots, S_k\}$, with $n > k \geq 1$, is a feasible set of non-singleton coalitions, i.e., for all $t, t' \in \{1, \dots, k\}$ such that $t \neq t'$, $S_t \cap S_{t'} = \emptyset$. Note that $\bigcup_{t=1}^k S_t \subset N$ may hold. This means that there may be players who are not in any of the specified non-singleton coalitions, in which case all $i \notin \bigcup_{t=1}^k S_t$ stay alone. For all

$\Pi^* \subseteq \Pi$, let $\Sigma(\Pi^*)$ denote the set of all coalition structures $\sigma = \{S_1, \dots, S_k\}$ such that for all $t \in \{1, \dots, k\}$, $S_t \in \Pi^*$.

For all $\Pi^* \subseteq \Pi$, let $\mathcal{R}(\Pi^*)$ denote the **set of preferences** over $\Pi^* \cup \{i\}$ for each player $i \in N$. Since we assume that players only care about the coalition they join, player i 's preferences $R_i \in \mathcal{R}(\Pi^*)$ *strictly* order all $S \in \Pi^* \cup \{i\}$ containing i , i.e., all the coalitions that player i is a member of. Preferences are strict, complete, and transitive. We will write $SP_i S'$ to indicate strict preferences, and $SR_i S'$ to indicate that either $SP_i S'$ or $S = S'$. A **(hedonic) coalition formation problem** is defined by a **preference profile** $R = (R_1, \dots, R_n) \in \mathcal{R}^n(\Pi)$, where $\mathcal{R}^n(\Pi) = \mathcal{R}(\Pi) \times \dots \times \mathcal{R}(\Pi)$ is the n -fold Cartesian product of $\mathcal{R}(\Pi)$. Given $R \in \mathcal{R}^n(\Pi)$ and coalition S , we denote $(R_i)_{i \in S}$ by R_S . For simplicity, we also write R_{-i} to denote $R_{N \setminus \{i\}}$, and R_{-S} to denote $R_{N \setminus S}$.

For a coalition formation model $\Pi^* \subset \Pi$, we restrict preferences to the coalitions that are feasible in a coalition formation model Π^* , and denote the set of restricted preference profiles by $\mathcal{R}^n(\Pi^*)$. For convenience, we assume that $R \in \mathcal{R}^n(\Pi^*)$ if and only if $R \in \mathcal{R}^n(\Pi)$ such that for all $S \in \Pi \setminus \Pi^*$, for all $i \in S$, $\{i\} P_i S$. We will say that \bar{R} is the **restriction of R to Π** if $\bar{R} \in \mathcal{R}^n(\Pi^*)$ such that for all $i \in N$ and $S_1, S_2 \in \Pi^*$, $S_1 \bar{R}_i S_2$ if and only if $S_1 R_i S_2$.

Given a coalition formation model $\Pi^* \subset \Pi$ and a preference profile $R \in \mathcal{R}^n(\Pi^*)$, a coalition S is a **blocking coalition of $\sigma \in \Sigma(\Pi)$ for R** if for all $i \in S$, $SP_i \sigma_i$. A coalition structure σ is **stable for $R \in \mathcal{R}^n(\Pi)$** if there does not exist a blocking coalition of σ for R . Note that the stable coalition structures are also the coalition structures in the core, since the two notions are identical in this context.

A **coalition formation problem $R \in \mathcal{R}^n(\Pi)$ is stable** if there exists a stable coalition structure for R . A **coalition formation model $\Pi^* \subseteq \Pi$ is stable** if for each coalition formation problem there exists a stable coalition structure, i.e., for all $R \in \mathcal{R}^n(\Pi^*)$, R is stable.

Given a coalition formation model Π^* and a coalition structure $\sigma \in \Sigma(\Pi^*)$, another coalition structure $\sigma' \in \Sigma(\Pi^*)$ is a **successor of σ** for $R \in \mathcal{R}^n(\Pi^*)$ if, given some blocking coalition $\bar{S} \in \Pi^* \cup [N]$ of σ for R , $\sigma' = \{S \in \Pi^* : S = \bar{S} \text{ if } |\bar{S}| \neq 1 \text{ or } S \in \sigma \text{ such that } S \cap \bar{S} \neq \emptyset\}$. Then σ' is the **\bar{S} -successor** of σ for R . Note that the \bar{S} -successor of σ is unique for each blocking coalition \bar{S} of σ .

Given a coalition formation model Π^* , a **coalition formation problem** $R \in \mathcal{R}^n(\Pi^*)$ is **myopically stable** if it is stable and, for all unstable coalition structures $\sigma_1 \in \Sigma(\Pi^*)$ there exists a finite sequence of coalition structures $(\sigma_1, \sigma_2, \dots, \sigma_k)$ for R such that for all $t \in \{1, \dots, k-1, \}$, σ_{t+1} is a successor of σ_t , and σ_k is stable for R . We will say in the following that σ_1 converges to σ_k for R , if there exists a sequence of coalition structures $(\sigma_1, \sigma_2, \dots, \sigma_k)$ for R such that for all $t \in \{1, \dots, k-1, \}$, σ_{t+1} is a successor of σ_t .

A **coalition formation model** Π^* is **myopically stable** if for all $R \in \mathcal{R}^n(\Pi^*)$, R is myopically stable. A coalition formation model Π^* is **conditionally myopically stable** if for all $R \in \mathcal{R}^n(\Pi^*)$, whenever R is stable, R is myopically stable. Note that if Π^* is myopically stable then it is both stable and conditionally myopically stable.

A coalition $S \in \Pi$ is **individually rational at R** if for all $i \in S$, $SP_i\{i\}$.⁵ For all $\Pi^* \subseteq \Pi$ and for all $R \in \mathcal{R}^n(\Pi^*)$, let

$$\Pi_{IR}(R) = \{S \in \Pi : \text{for all } i \in S, SP_i\{i\}\}$$

be the **set of individually rational coalitions at R** . Note that if σ is stable at R then $\sigma \subseteq \Pi_{IR}(R)$.

3 Graph Representation of Hedonic Coalition Formation

The following graph representations, which were introduced in Pápai (2002), provide a very useful tool for our analysis. We use directed graphs to represent the preferences over

⁵For convenience, we exclude singletons from this definition as well.

individually rational coalitions for a coalition formation problem R . In addition, we use undirected graphs to describe the relations of feasible coalitions in a coalition formation model Π^* (what we refer to as a *coalition configuration*).

First we recall the necessary basic concepts about directed and undirected graphs. A *directed graph* $\vec{\Gamma} = (V, D)$ is given by a set D of ordered pairs of distinct vertices in V . An ordered pair $(v, z) \in D$ is called a *directed edge* from v to z , where $v, z \in V$. A *directed path* in $\vec{\Gamma}$ is a sequence of edges $\{(v_1, v_2), \dots, (v_{k-1}, v_k)\}$, such that all vertices v_1, \dots, v_k are distinct. Given such a directed path, we say that there is a path from v_1 to v_k via vertices $v_1, v_2, \dots, v_{k-1}, v_k$. If k is even then it is an *odd directed path* and if k is odd then it is an *even directed path*. If, in addition, $(v_k, v_1) \in D$, the sequence of directed edges form a **directed general cycle** $\{(v_1, v_2), \dots, (v_k, v_1)\}$. A **directed cycle** is a chordless cycle; that is, for any two distinct vertices v, z in the cycle, if there is no directed edge from v to z in the cycle, then there is no directed edge from v to z in the graph. Furthermore, if k is even then it is an **even directed cycle** and if k is odd then it is an **odd directed cycle**.

In the following, we will use the notation \vec{vz} to indicate that there is an edge from v to z . We will say that if \vec{vz} or \vec{zv} in $\vec{\Gamma}$ then v and z are *neighbors* in $\vec{\Gamma}$. We are also going to say that two disjoint vertex sets $V_1, V_2 \subset V$ are *neighbors* in $\vec{\Gamma}$ if there exist $v_1 \in V_1$ and $v_2 \in V_2$ such that v_1 and v_2 are neighbors in $\vec{\Gamma}$.

Fix a coalition formation model Π^* and fix $R \in \mathcal{R}^n(\Pi^*)$. We construct the graph $\vec{\Gamma}_{\Pi^*}(R)$ to represent the coalition formation problem as follows. $\vec{\Gamma}_{\Pi^*}(R) = (\Pi_{IR}^*(R), D)$, that is, the vertex set of $\vec{\Gamma}_{\Pi^*}(R)$ is the set of individually rational coalitions at R . The set of directed edges D is determined in the following manner. For all $S_1, S_2 \in \Pi_{IR}^*(R)$, $\vec{S_1S_2}$ in $\vec{\Gamma}_{\Pi^*}(R)$ if and only if there exists $i \in S_1 \cap S_2$ such that $S_2 P_i S_1$. Note that the undirected graph associated with $\vec{\Gamma}_{\Pi^*}(R)$ is the intersection graph of $\Pi_{IR}^*(R)$, that is, for all $S_1, S_2 \in \Pi_{IR}^*(R)$ such that $S_1 \neq S_2$, there is an edge from S_1 to S_2 or from S_2 to S_1 if $S_1 \cap S_2 \neq \emptyset$. From now on, we may refer to R and $\vec{\Gamma}_{\Pi^*}(R)$ interchangeably. For example,

we may say that R contains a directed cycle, instead of $\vec{\Gamma}_{\Pi^*}(R)$.

In the lemma below, as in Pápai (2002), we present a simple way of identifying stable coalition structures in a graph representing a coalition formation problem $R \in \mathcal{R}^n(\Pi^*)$.

Lemma 1 Fix $R \in \mathcal{R}^n(\Pi)$. Let $\bar{\Pi} \subseteq \Pi_{IR}(R)$ such that it satisfies

- (1) *feasibility*: for all $S_1, S_2 \in \bar{\Pi}$, S_1 and S_2 are not neighbors in $\vec{\Gamma}_{\Pi}(R)$, and
- (2) *non-blocking*: for all $S_1 \in \Pi_{IR}(R) \setminus \bar{\Pi}$, there exists $S_2 \in \bar{\Pi}$ such that $\overrightarrow{S_1 S_2}$ in $\vec{\Gamma}_{\Pi}(R)$.

Then $\bar{\Pi}$ is a stable coalition structure for R .

Conversely, let $\sigma \in \Sigma(\Pi)$ be a stable coalition structure for R . Then σ satisfies both *feasibility* and *non-blocking*.

In the lemma, *feasibility* requires that if two non-singleton coalitions are in a stable coalition structure then there is no directed edge from one to the other. This means that the two coalitions do not contain a common member, which is indeed necessary, since otherwise it wouldn't be feasible to form the two coalitions simultaneously. Furthermore, *non-blocking* says that every individually rational coalition that is not in the given stable coalition structure has a directed edge pointing to a coalition that is in this stable coalition structure. This means that there is a member of coalition S , where S is not in this stable coalition structure, such that this member strictly prefers his coalition in the stable coalition structure to S , and thus S is not a blocking coalition.

In the following, we may refer to $\bar{\Pi} \subset \Pi^*$ as a **stable solution** of $\vec{\Gamma}_{\Pi^*}(R)$, if it satisfies both *feasibility* and *non-blocking*, as defined in Lemma 1. A stable solution of $\vec{\Gamma}_{\Pi^*}(R)$ always satisfies *feasibility* in Lemma 1. If $\bar{\Pi}$ is a stable solution, we may refer to it as a feasible set. In general, $\bar{\Pi} \subset \Pi$ is a *feasible set* if for all $S_1, S_2 \in \bar{\Pi}$, S_1 and S_2 are not neighbors in $\vec{\Gamma}_{\Pi^*}(R)$, i.e., $S_1 \cap S_2 = \emptyset$. A feasible set $\bar{\Pi}$ is *maximal* for $\vec{\Gamma}_{\Pi^*}(R)$ if there is no superset $\hat{\Pi} \supset \bar{\Pi}$ such that $\hat{\Pi}$ is also a feasible set for $\vec{\Gamma}_{\Pi^*}(R)$.

An undirected graph $\Gamma = (V, E)$ consists of a set of vertices V and a set of undirected

edges E between pairs of vertices in V . In a multigraph, multiple edges between two vertices are allowed. For each coalition formation model Π^* , we construct the undirected multigraph Γ_{Π^*} to represent Π^* . $\Gamma_{\Pi^*} = (\Pi^*, E)$, that is, the vertex set is the set of non-singleton coalitions in Π^* . The edge set, E , consists of undirected edges, such that the maximum number of multiple edges is two. We denote an edge between vertices v and z by $e_1(v, z)$, and, if there is a second edge between v and z , we denote it by $e_2(v, z)$. Given Π^* , we construct Γ_{Π^*} as follows. Let $S_1, S_2 \in \Pi^*$ such that $S_1 \neq S_2$. Then there exists $e_1(S_1, S_2) \in E$ if and only if $S_1 \cap S_2 \neq \emptyset$, and there exists $e_2(S_1, S_2) \in E$ if and only if $|S_1 \cap S_2| \geq 2$.

Just as for directed graphs, we say that two vertices or sets of vertices are *neighbors* in an undirected graph if there is an edge between them. For undirected graphs, the definitions concerning paths are similar to the definitions given for directed graphs, except that the edges are not directed. Moreover, a collection of coalitions $\hat{\Pi} \subset \Pi$ is a **cycle** if $\hat{\Pi} = \{S_1, \dots, S_k\}$, where S_1, \dots, S_k are distinct coalitions, $k \geq 3$, and $\hat{\Pi}$ satisfies the following properties:

- (i) for all $t \in \{1, \dots, k-1\}$, $S_t \cap S_{t+1} \neq \emptyset$, and $S_1 \cap S_k \neq \emptyset$,
- (ii) for all other pairs S_t, S_l , $S_t \cap S_l = \emptyset$,
- (iii) if $k = 3$, $S_1 \cap S_2 \cap S_3 = \emptyset$.

We say that $\Gamma_{\hat{\Pi}}$ is a cycle if it represents a set of feasible coalitions $\hat{\Pi}$ such that $\hat{\Pi}$ is a cycle. A collection of coalitions $\hat{\Pi}$ and, similarly, $\Gamma_{\hat{\Pi}}$, is an **odd cycle** if it is a cycle $\hat{\Pi} = \{S_1, \dots, S_k\}$ and k is odd, and it is an **even cycle** if k is even.

Fix an undirected graph $\Gamma = (V, E)$. Then an *induced subgraph* of Γ is given by a graph $\Gamma' = (V', E')$, where $V' \subseteq V$, and for all $v, z \in V'$, $e_1(v, z) \in E'$ if and only if $e_1(v, z) \in E$, and $e_2(v, z) \in E'$ if and only if $e_2(v, z) \in E$. A graph is *connected* if for all $v, z \in V$, there is a path between v and z . The *connected components* of a graph are induced subgraphs, such that for all v, z in a connected component there is a path between v and z , and for all

v and z in different connected components there is no path between v and z . A directed graph is *strongly connected* if for all pairs of vertices $v, z \in V$, there is a directed path from v to z .

For any pair of distinct vertices $v, z \in V$, if v and z are in a connected component in Γ then there exists a path in Γ which connects v and z . The path which contains the least number of edges among those connecting v and z is called a *shortest path* between v and z . The length of a shortest path is called the *distance* between two vertices. If the shortest path between two vertices is an odd path, then there is an *odd distance* between them, and if it is even, then there is an *even distance* between them.

Let $\Gamma_C = (V, E)$ be an odd cycle. Let $\Gamma_{C_-} = (V, E')$ be such that for all $S_1, S_2 \in C$, $e_1(S_1, S_2) \in E'$ if and only if $e_1(S_1, S_2) \notin E$. Then Γ_{C_-} is an **odd anticycle**. In order to define an odd directed anticycle, let Γ_{C_-} be an odd anticycle. Then the vertices in C_- can be numbered from 1 to k (where $k \geq 7$ is odd)⁶ such that each vertex S_t is a neighbor of the following vertices only: $S_{t+1}, S_{t+2}, \dots, S_{t+\frac{k-3}{2}}, S_{t-1}, S_{t-2}, \dots, S_{t-\frac{k-3}{2}}$, where $t \in \{1, \dots, k\}$ and $S_{t+1} := S_1$. Then $\vec{\Gamma}_{C_-}$ is an **odd directed anticycle** if for each $t \in \{1, \dots, k\}$, $\overrightarrow{S_t S_{t+1}}, \dots, \overrightarrow{S_t S_{t+\frac{k-3}{2}}}$ hold, and $\overrightarrow{S_{t+1} S_t}, \dots, \overrightarrow{S_{t+\frac{k-3}{2}} S_t}$ do not hold in $\vec{\Gamma}_{C_-}$.

Let $W = \{S_1, \dots, S_k\} \subset \Pi$. W is a **pairwise cycle** if for all $t, l \in \{1, \dots, k\}$, $S_t \cap S_l \neq \emptyset$, and $\bigcap_{t=1}^k S_t = \emptyset$. $\vec{\Gamma}_W$ is a **directed pairwise cycle** if for all $t, l \in \{1, \dots, k\}$ such that $t < l$, $\overrightarrow{S_t S_l}$ in $\vec{\Gamma}_W$, except $\vec{\Gamma}_W$ does not have $\overrightarrow{S_l S_t}$ if $t + 1 = l$, where $S_{k+1} := S_1$.

4 A Sufficient Condition for the Myopic Stability of Coalition Formation Problems

We are going to present a preliminary result on the myopic stability of hedonic coalition formation problems. This result is interesting in its own right, and will also be used later to

⁶If $k = 3$, then Γ_{C_-} is a set of isolated vertices; if $k = 5$, then $\Gamma_{C_-} = \Gamma_C$.

prove our main characterization theorem. The proof of the proposition is in the Appendix.

Proposition 1 *If a coalition formation problem $R \in \mathcal{R}^n(\Pi)$ does not contain an odd directed cycle, an odd directed anticycle, or a pairwise directed cycle then it is myopically stable.*

This proposition on the more general hedonic coalition formation generalizes Chung's related result (Lemma 1, Chung (2000)) on the roommate model. It is important to note, however, that his result holds for weak preferences as well, whereas ours is only proved for strict preferences. The Chung (2000) lemma requires that we prohibit odd directed cycles. In the roommate model, odd directed anticycles and pairwise directed cycles cannot occur, since the coalition size is maximum two. This explains why these two results correspond to each other, assuming that preferences are strict.

5 The Main Characterization Theorem on Coalition Formation Models

For all odd cycles $C \subset \Pi$, let Ω_C denote the set of collections of coalitions $\bar{\Pi} \subseteq C$ such that $|\bar{\Pi}| = 3$ and the three coalitions in $\bar{\Pi}$ are an odd distance away from each other in C . For all odd anticycles $C_- \subset \Pi$, let Ω_{C_-} denote the set of collections of coalitions $\bar{\Pi}$ such that $|\bar{\Pi}| = \frac{k+1}{2}$, given that C_- is a k -anticycle and $k \geq 7$ is odd, and the $\frac{k+1}{2}$ coalitions in $\bar{\Pi}$ form a path in the main cycle of C_- . For all pairwise cycles $W \subset \Pi$, let $\Omega_W = \{W\}$. Let G be an odd cycle, odd anticycle, or pairwise cycle. Then Ω_G will denote Ω_C , Ω_{C_-} , or Ω_W , whichever is appropriate.

A set of coalitions $\hat{\Pi} \subset \Pi$ has a **minimal inaccessible coalition configuration** if the induced subgraph $\Gamma_{\hat{\Pi}}$ contains an odd cycle, odd anticycle, or pairwise cycle, which we denote by G , and if $\Gamma_{\hat{\Pi}}$ satisfies one of the two cases below. (Examples of $\Gamma_{\hat{\Pi}}$ of the two

cases are presented in Figures 1 and 2, respectively.⁷⁾

[FIGURE 1 HERE]

Case 1 (“Handcuffs”): $\Gamma_{\hat{\Pi}}$ consists of G , an even cycle \hat{C} which is disjoint from G , and coalitions in a path connecting \hat{C} and G . Let this path be given by $\{e_1(S_1, S_2), \dots, e_2(S_{k-1}, S_k)\}$, with $k \geq 2$, such that S_1 is in \hat{C} , S_k is in G , and one of the following two subcases holds:

Subcase 1a: If $k > 2$, then S_2 is the only neighbor of \hat{C} , S_{k-1} is the only neighbor of G , and for all $S \in \hat{C} \setminus \{S_1\}$ that is a neighbor of S_2 , S is an even distance away in \hat{C} from S_1 .

Subcase 1b: If $k = 2$, then all $S \in \hat{C} \setminus \{S_1\}$ that is a neighbor of G is an even distance away in \hat{C} from S_1 .

[FIGURE 2 HERE]

Case 2 (“Cages”):

Let $\bar{\Pi} = \{S \in \hat{\Pi} \setminus G : \text{there exists } \bar{\bar{\Pi}} \in \Omega_G \text{ such that for all } S' \in \bar{\bar{\Pi}}, S \text{ and } S' \text{ are neighbors in } \Gamma_{\hat{\Pi}}\}$.

There exists $\Pi' \subseteq \bar{\Pi}$ ($\Pi' \neq \emptyset$), for all $S \in \Pi'$, there exists $\bar{\bar{\Pi}}_S \in \Omega_G$, and there exists $\tilde{\Pi} \in \Omega_G$ such that the following conditions hold:

- (1) For all $S \in \Pi'$ and for all $S' \in \bar{\bar{\Pi}}_S$, S and S' are neighbors in $\Gamma_{\hat{\Pi}}$.
- (2) Let $\Gamma'_{\hat{\Pi}}$ be the same as $\Gamma_{\hat{\Pi}}$, except: for all $S \in \Pi'$ and $S' \in \bar{\bar{\Pi}}_S$, omit one edge between S and S' . For all $\tilde{S} \in \tilde{\Pi}$ there exists $S \in \Pi'$, and for all $S \in \Pi'$ there exists $\tilde{S} \in \tilde{\Pi}$ for which

⁷For each graph in these figures, if there is only one edge between two vertices then a similar graph with two edges between these two vertices also works; however, if there are two edges indicated between two vertices then a graph with one edge between these vertices does not work as an example of a minimal inaccessible coalition configuration.

the following hold: there is a cycle in $\Gamma_{\hat{\Pi}}$ that contains S and \tilde{S} such that one of the two paths in this cycle that connects S and \tilde{S} is an odd path and is contained in $\Gamma'_{(\hat{\Pi} \setminus G) \cup \{\tilde{S}\}}$.

(3) $\hat{\Pi}$ consists of G , Π' , and coalitions in the odd paths specified in (2).

Finally, the following conditions have to be satisfied when G is an odd cycle or odd anticycle, respectively.

Let $\check{\Pi} = \{\tilde{S} \in G : \text{there exists } S \in \hat{\Pi} \setminus G \text{ such that } \tilde{S} \text{ and } S \text{ are neighbors in } \Gamma'_{\hat{\Pi}}\}$.

(A) If G is an odd cycle then for all $S \in G \setminus \check{\Pi}$, call S *odd-signed* if S is an odd distance away in G from the closest coalition $\check{S} \in \check{\Pi}$, when traversing G clockwise, and call S *even-signed* otherwise. Then, if an odd-signed coalition in $G \setminus \check{\Pi}$ has a neighbor outside of G in $\hat{\Pi}$, then no even-signed coalition in $\check{\Pi}$ has a coalition outside of G in $\hat{\Pi}$. Similarly, if an even-signed coalition in $G \setminus \check{\Pi}$ has a neighbor outside of G in $\hat{\Pi}$, then no odd-signed coalition in $\check{\Pi}$ has a neighbor outside of G .

(B) If G is an odd anticycle, then for all $S \in G \setminus \check{\Pi}$, call S *first-signed* if S is the closest to a coalition $\check{S} \in \check{\Pi}$, when traversing the main cycle of G clockwise, and call $S \in G \setminus \check{\Pi}$ *last-signed* if S is the closest to a coalition $\check{S} \in \check{\Pi}$, when traversing the main cycle of G counter-clockwise. Then either the first-signed or the last-signed coalition has no neighbor outside of G .

We will call $\Gamma_{\hat{\Pi}}$ a **minimal inaccessible coalition configuration**. We say that $\Pi^* \subseteq \Pi$ has an **inaccessible coalition configuration** if there exists $\bar{\Pi} \subseteq \Pi^*$ such that it has a minimal inaccessible coalition configuration.

Now we can state our main result which characterizes conditionally myopically stable coalition formation models. Given the above definition, the Main Theorem describes conditional myopic stability in terms of forbidden subgraphs (the minimal inaccessible coalition configurations) in the undirected graph of a coalition formation model.

Main Theorem *A coalition formation model $\Pi^* \subseteq \Pi$ is conditionally myopically stable if and only if it does not have an inaccessible coalition configuration.*

The proof is in the Appendix.

6 Implications for Myopic Stability

First we present corollaries to the Main Theorem that concern the myopic stability of a coalition formation model.

Corollary 1 *If a coalition formation model is stable then it is myopically stable.*

This is an immediate implication of the Main Theorem. If a coalition formation model is stable then it cannot allow for an odd cycle, odd anticycle, or pairwise cycle. However, each minimal inaccessible coalition configuration contains an odd cycle, odd anticycle, or pairwise cycle. Thus, a stable coalition formation model does not have an inaccessible coalition configuration, and the Main Theorem implies that any such model is myopically stable. In fact, Proposition 1 also implies Corollary 1.

Corollary 1 is an important result since it shows that stability and myopic stability are identical requirements for a coalition formation model. In addition, one can simply verify the stability of a coalition formation model (see Pápai (2002)), which is easier than checking myopic stability, and be assured that the model is myopically stable.

Another immediate corollary is the Roth and Vande Vate (1990) result regarding the marriage model, but our main theorem only implies their result for strict preferences. Before stating the corollary, let us recall that in the marriage model the players are partitioned into two sets, the set of “men” \mathcal{M} and “women” \mathcal{W} , and each non-singleton coalition consists of a “man” and a “woman.” Formally, $\Pi^* = \{S \subseteq N : \text{there exist } \mathcal{M} \subset N \text{ and } \mathcal{W} \subset N \text{ such that } \mathcal{M} \cup \mathcal{W} = N, \mathcal{M} \cap \mathcal{W} = \emptyset, \text{ and } S = \{m, w\}, \text{ where } m \in \mathcal{M} \text{ and } w \in \mathcal{W}\}$.

Corollary 2 [Roth and Vande Vate (1990)] *The marriage model is myopically stable.*

This corollary simply follows from Corollary 1, since the marriage model is stable (Gale and Shapley, 1962).

7 Implications for Conditional Myopic Stability

While myopic stability holds automatically for any stable coalition formation model, coalition formation models that are not stable may also satisfy myopic stability in the necessarily limited sense that whenever a preference profile is stable, it is myopically stable. That is, a coalition formation model which is not stable may satisfy conditional myopic stability. One notable example of this is the roommate model, which was proved by Diamantoudi, Miyagawa, and Xue (2002).

We need the following definitions for our next corollary. A coalition formation model Π^* contains a **multiple overlap** if there exist distinct coalitions $S, S' \in \Pi^*$ such that $|S \cap S'| \geq 2$. A coalition formation model Π^* contains an **open claw** if there exist distinct coalitions $S, S_1, S_2, S_3 \in \Pi^*$ such that for $t = \{1, 2, 3\}$, $S \cap S_t \neq \emptyset$ and for all $t, l \in \{1, 2, 3\}$ such that $t \neq l$, $S_t \cap S_l = \emptyset$. A coalition formation model Π^* contains a **triangle claw** if there exist distinct coalitions $S, S_1, S_2, S_3 \in \Pi^*$ such that for $t = \{1, 2, 3\}$, $S \cap S_t \neq \emptyset$ and S_1, S_2, S_3 form an odd cycle.

Corollary 3 *If a coalition formation model does not contain a multiple overlap, an open claw, or a triangle claw, then it is conditionally myopically stable.*

It is tedious but straightforward to verify that if a coalition formation model Π^* has an inaccessible coalition configuration then it contains at least one multiple overlap, open claw, or triangle claw. Thus, the above corollary follows from the Main Theorem.

Recall that a roommate model is given by $\Pi^* = \{S \subseteq N : |S| = 2\}$, i.e., each feasible non-singleton coalition has two members.

Corollary 4 [Diamantoudi, Miyagawa, and Xue (2002)] *The roommate model is conditionally myopically stable.*

The above result follows from Corollary 3. Indeed, the roommate model does not contain a multiple overlap since each non-singleton coalition is of size two, and therefore the overlap between two distinct coalitions is at most one. The roommate model does not contain an open claw either since, given $S = \{i, j\}$, we can have player i in coalition S_1 , player j in coalition S_2 , and there is no player left for coalition S_3 who is in S , but not in S_1 or S_2 . It is also easy to see that the roommate model does not contain a triangle claw. Given an odd 3-cycle $S_1 = \{1, 2\}$, $S_2 = \{2, 3\}$, $S_3 = \{1, 3\}$, if S is to be a neighbor of all three coalitions in the triangle, it has to have two players out of 1, 2 and 3, since there is no common member in all three coalitions. This would mean that S is not distinct from one of S_1 , S_2 , or S_3 , implying that the roommate model does not contain a triangle claw.

8 Conclusion

One interesting extension of our work would be to check for the robustness of the result in a stochastic framework. While our result shows when a myopic blocking sequence converges to a stable coalition structure in a hedonic coalition formation model with probability one, the process is deterministic, and the result is essentially an existence result. Jackson and Watts (2002) examine a stochastic dynamic process for network formation, which is similar to our approach in that they study a myopic process, but refine it by adding a stochastic part to the analysis. They apply this analysis to the marriage model and the college admission model, and find that all core stable networks are stochastically stable. If the dynamic

solution concept of Jackson and Watts (2002) were adopted to hedonic coalition formation models, perhaps making use of hypergraphs, the predictions obtained would be very useful for applications in which we would expect some noise.

Appendix

Proof of Proposition 1

Step 1: Fix $R \in \mathcal{R}^n$ such that R does not contain an odd directed cycle, an odd directed anticycle, or a pairwise directed cycle. Suppose R is not myopically stable. Then there exists a directed general cycle $\vec{G} = \{(S_1, S_2), \dots, (S_{v-1}, S_v), (S_v, S_1)\}$, where, for $l \in \{1, \dots, v\}$, $S_l \in \Pi$, for which there exists a sequence of unstable coalition structures $\sigma_1, \dots, \sigma_k$ such that for all $t \in \{1, \dots, k-1\}$, σ_{t+1} is a successor of σ_t (where $\sigma_{k+1} := \sigma_1$), and for all $l \in \{1, \dots, v\}$, there exists $t \in \{1, \dots, k\}$ such that σ_{t+1} is an S_l -successor of σ_t with respect to R . Let $\Pi^* = \{S_1, \dots, S_v\}$, and let R^* be the restriction of R to Π^* . For all $t \in \{1, \dots, k\}$, let $\sigma_t^* = \sigma_t \cap \Pi^*$. Note that there may exist $t \in \{1, \dots, k\}$ such that $\sigma_t^* = \sigma_{t+1}^*$. Renumber σ_t^* 's accordingly: let $\sigma_{1^*}, \dots, \sigma_{k^*}$ (where $k^* \leq k$) such that $\sigma_{1^*}^* = \sigma_1^*$, $\sigma_{2^*}^* = \sigma_t^*$, where for all t' such that $1 < t' < t$, $\sigma_{t'^*}^* = \sigma_1^*$, if any, and $\sigma_t^* \neq \sigma_1^*$, etc., so that for all $t \in \{1^*, \dots, k^*-1\}$, $\sigma_t^* \neq \sigma_{t+1}^*$. Note that if σ_{t+1} is an S_l -successor of σ_t with respect to R , it means that there is no $S \in \sigma_t$ such that $\overrightarrow{S_l S}$ in $\vec{\Gamma}_\Pi(R)$. Then, for all $t \in \{1^*, \dots, k^*-1\}$, σ_{t+1}^* is a successor of σ_t^* (where $\sigma_{k^*+1}^* := \sigma_{1^*}^*$), and for all $t \in \{1^*, \dots, k^*\}$, there exists $l \in \{1, \dots, v\}$ such that σ_{t+1}^* is an S_l -successor of σ_t^* for R^* .

Without loss of generality, we can assume that \vec{G} is minimal in the sense that there exists no \vec{H} such that $\vec{H} = \{(S'_1, S'_2), \dots, (S'_{v'-1}, S'_{v'}), (S'_{v'}, S'_1)\}$, where $\{S'_1, \dots, S'_{v'}\} \subset \{S_1, \dots, S_v\}$, $\Pi' = \{S'_1, \dots, S'_{v'}\}$, and R^* restricted to Π' is not myopically stable. We will refer to this in the following as the minimality of \vec{G} . Then, for all $S \in \Pi^*$, it is not the case

that for all $t \in \{1^*, \dots, k^*\}$, $S \in \sigma_t^*$, since otherwise \vec{G} would not be minimal. Similarly, for all $S \in \Pi^*$, it is not the case that for all $t \in \{1^*, \dots, k^*\}$, $S \notin \sigma_t^*$.

Step 2: Since $\vec{\Gamma}_\Pi(R)$ does not contain an odd directed cycle, an odd directed anticycle, or a pairwise directed cycle, and since R^* is the restriction of R to Π^* , it follows that $\vec{\Gamma}_{\Pi^*}(R^*)$ does not contain an odd directed cycle, an odd directed anticycle, or a pairwise directed cycle. Therefore, Pápai (2002) implies that R^* is stable. Fix a stable coalition structure at R^* , and fix $l \in \{1, \dots, v\}$ such that S_l is in the fixed stable coalition structure at R^* . Then *feasibility* in Lemma 1 implies that S_{l+1} is not in the stable coalition structure at R^* . Thus, either S_{l+2} is in the stable coalition structure at R^* , or *non-blocking* in Lemma 1 implies that there exists $l' \in \{1, \dots, v\}$ such that $\overrightarrow{S_{l+1}S_{l'}}$ in $\vec{\Gamma}_{\Pi^*}(R^*)$, $S_{l'}$ is in the fixed stable coalition structure at R^* , and there is no l'' such that $l+2 < l'' < l'$, $\overrightarrow{S_{l+1}S_{l''}}$ in $\vec{\Gamma}_{\Pi^*}(R^*)$, and $S_{l''}$ is in the stable coalition structure at R^* . If we keep continuing, we will find an even directed general cycle $\vec{C} = \{(S_{i_1}, S_{i_2}), \dots, (S_{i_{m-1}}, S_{i_m}), (S_{i_m}, S_{i_1})\}$ in $\vec{\Gamma}_{\Pi^*}(R^*)$ such that $i_t < i_{t+1}$, where $i_{m+1} := i_1$, and for all even $t \leq m$, S_{i_t} is in the stable coalition structure at R^* . Let $\bar{C} := \{S_{i_t} : t \leq m \text{ is even}\}$ be the set of coalitions in \vec{C} that are in the stable coalition structure at R^* . Note that if $l \in \{1, \dots, v\}$ is such that S_l is in \bar{C} and S_{l+1} is not in \bar{C} then $S_l \notin \bar{C}$. In the following, we will refer to a coalition which is in \vec{C} but not in \bar{C} as a *cut-off coalition*.

Call $\overrightarrow{S_1S_2}, \dots, \overrightarrow{S_{v-1}S_v}, \overrightarrow{S_vS_1}$ the *main cycle* of \vec{G} , and call $\overrightarrow{S_{i_1}S_{i_2}}, \dots, \overrightarrow{S_{i_{m-1}}S_{i_m}}, \overrightarrow{S_{i_m}S_{i_1}}$ the *main cycle* of \vec{C} .

Step 3: Fix $t \in \{1^*, \dots, k^*\}$. Suppose $\bar{C} \subseteq \sigma_t^*$. In other words, suppose that every even indexed coalition in \vec{C} is in σ_t^* . We will show that this leads to a contradiction. First, notice that this implies that every blocking coalition for σ_t^* with respect to R^* is in a cut-off coalition. Then, since \vec{G} is minimal, there exist S, S' in \vec{G} such that:

(i) $S \notin \bar{C}$, $S' \in \bar{C}$, and $\overrightarrow{SS'}$ in the main cycle of \vec{C} ,

(ii) there is a directed path to S in the main cycle of \vec{G} from a cut-off coalition which is a blocking coalition for σ_t^* with respect to R^* ,

(iii) there exists $t' > t$ such that S is a blocking coalition for $\sigma_{t'}^*$ with respect to R^* , and there is no t'' , with $t < t'' < t'$, such that \vec{C} contains a blocking coalition for $\sigma_{t''}^*$ with respect to R^* .

Thus, since \vec{G} is minimal and there is no t'' , with $t < t'' < t'$, such that \vec{C} contains a blocking coalition for $\sigma_{t''}^*$ with respect to R^* , and given that $S' \in \vec{C}$, so that there is no directed edge in the main cycle of \vec{G} to a cut-off coalition, $S' \in \sigma_t^*$ implies that $S' \in \sigma_{t'}^*$. Since $\vec{S}\vec{S}'$ in \vec{G} , this means that S is not a blocking coalition for $\sigma_{t'}^*$ with respect to R^* , which is a contradiction. Therefore, for all $t \in \{1^*, \dots, k^*\}$, $\vec{C} \not\subseteq \sigma_t^*$.

Step 4: Fix $S \in \vec{C}$. Then, since \vec{G} is minimal, there exists $t \in \{1, \dots, k\}$ such that $S \in \sigma_t^*$. Fix $t \in \{1^*, \dots, k^*\}$ such that $S \in \sigma_t^*$. We will prove that there exists \vec{S} in \vec{G} such that \vec{S} is a blocking coalition for σ_t^* with respect to R^* , and we don't have $\vec{S}\vec{S}$ in $\vec{\Gamma}_{\Pi^*}(R^*)$. As shown in Step 3, $\vec{C} \not\subseteq \sigma_t^*$. Since $S \in \vec{C} \cap \sigma_t^*$ and \vec{C} is even, there exist at least two coalitions in \vec{C} which are next to each other in the main cycle of \vec{C} such that neither coalition is in σ_t^* . Going backwards (in the opposite direction of the directed edges) from S in the main cycle of \vec{C} , we can therefore find a coalition \vec{S} in \vec{C} such that it is the first coalition among the traversed coalitions that follows another coalition, say \hat{S} , (i.e., $\vec{S}\vec{\hat{S}}$ in the main cycle of \vec{C}) such that $\vec{S} \notin \sigma_t^*$ and $\hat{S} \notin \sigma_t^*$. Note that since two coalitions that are neighbors cannot both be in σ_t^* , by *feasibility* in Lemma 1, and therefore the way we identified \vec{S} and \hat{S} shows that $\vec{S} \in \vec{C}$ and $\hat{S} \notin \vec{C}$. Then, given $S \in \vec{C}$, it follows from *feasibility* of Lemma 1 that we don't have $\vec{S}\vec{S}$ in $\vec{\Gamma}_{\Pi^*}(R^*)$. We will show that \vec{S} is a blocking coalition for σ_t^* with respect to R^* .

Suppose there exists $S' \neq \hat{S}$ in \vec{G} such that $\vec{S}\vec{S}'$ in $\vec{\Gamma}_{\Pi^*}(R^*)$ and $S' \in \sigma_t^*$. Given that $\hat{S} \notin \sigma_t^*$, and given that \vec{G} is minimal, there exists $t' \in \{1^*, \dots, k^*\}$, $t' > t$, such that for

all such S' , $S' \notin \sigma_{t'}^*$, $\hat{S} \notin \sigma_{t'}^*$, \bar{S} is a blocking coalition for $\sigma_{t'}^*$, however, \bar{S} is not a blocking coalition for σ_t^* with respect to R^* for all l such that $t \leq l < t'$. Then, given that $\overrightarrow{\bar{S}\hat{S}}$ in the main cycle of \overrightarrow{G} , it follows that \overrightarrow{G} is not minimal, a contradiction. Therefore, for all $S' \neq \hat{S}$ in \overrightarrow{G} such that $\overrightarrow{\bar{S}S'}$ in $\overrightarrow{\Gamma}_{\Pi^*}(R^*)$, $S' \notin \sigma_t^*$. Thus, \bar{S} is a blocking coalition for σ_t^* with respect to R^* , as desired.

Step 5: In sum, we proved that there exists $S \in \bar{C}$ such that for all $t \in \{1^*, \dots, k^*\}$ for which $S \in \sigma_t^*$, there exists $\bar{S} \in \bar{C}$ such that \bar{S} is a blocking coalition for σ_t^* with respect to R^* . Then, by the feasibility of \bar{C} , we have $S \cap \bar{S} = \emptyset$, so that $S \in \sigma_{t+1}^*$. By repeating this argument iteratively, we get that for all $t \in \{1^*, \dots, k^*\}$, $S \in \sigma_t^*$. This contradicts the minimality of \overrightarrow{G} . Therefore, $R \in \mathcal{R}^n$ is myopically stable.

Proof of the Main Theorem

Before proving the theorem, we prove some useful lemmas.

Lemma 2 *Let $R \in \mathcal{R}^n(\Pi^*)$ be a coalition formation problem that is not myopically stable. Let $\bar{\Pi} \subseteq \Pi^*$ be such that for all coalition structures $\sigma \in \Sigma(\Pi^*)$ that do not converge to a stable coalition structure for R , if S is a blocking coalition of σ for R then $S \in \bar{\Pi}$. Let \bar{R} be the restriction of R to $\bar{\Pi}$. Then \bar{R} contains an odd directed cycle, odd directed anticycle, or directed pairwise cycle.*

Proof: Let $R \in \mathcal{R}^n(\Pi^*)$, $\bar{\Pi}$, and \bar{R} be as specified above. Note first that if $S \in \bar{\Pi}$ is a blocking coalition of σ for R , then Lemma 1 implies that there is no $S' \in \sigma$ such that $\overrightarrow{\bar{S}S'}$ in $\overrightarrow{\Gamma}_{\Pi^*}(R)$. Then it follows that there is no $S' \in \sigma \cap \bar{\Pi}$ such that $\overrightarrow{\bar{S}S'}$ in $\overrightarrow{\Gamma}_{\bar{\Pi}}(\bar{R})$. Therefore, S is a blocking coalition of $\sigma \cap \bar{\Pi}$ for \bar{R} . Thus $\sigma \cap \bar{\Pi} \in \Sigma(\bar{\Pi})$ does not converge to a stable coalition structure for \bar{R} , which means that \bar{R} is not myopically stable. Then Proposition 1 implies that \bar{R} contains an odd directed cycle, odd directed anticycle, or directed pairwise

cycle. \square

Lemma 3 *If there are at least two different stable coalition structures for a coalition formation problem $R \in \mathcal{R}^n(\Pi)$ then it contains an even directed cycle such that the two alternate stable solutions of the even directed cycle are in two different stable coalition structures for R .*

Proof: Fix $R \in \mathcal{R}^n(\Pi^*)$ such that it has two different stable coalition structures $\sigma_1, \sigma_2 \in \Sigma(\Pi^*)$. Let $\bar{\Pi}_1 = \{S \in \Pi^* : S \in \sigma_1, S \notin \sigma_2\}$ and let $\bar{\Pi}_2 = \{S \in \Pi^* : S \in \sigma_2, S \notin \sigma_1\}$. Let $\bar{\Pi} = \bar{\Pi}_1 \cup \bar{\Pi}_2$. Note that $\bar{\Pi}_1 \cap \bar{\Pi}_2 = \emptyset$, and thus $\bar{\Pi}_1$ and $\bar{\Pi}_2$ partition $\bar{\Pi}$. By *feasibility* in Lemma 1, for all $S_1, S_2 \in \bar{\Pi}$ such that there is an edge between S_1 and S_2 in $\Gamma_{\bar{\Pi}}$, $S_1 \in \bar{\Pi}_1$ and $S_2 \in \bar{\Pi}_2$, or vice versa. Therefore, $\Gamma_{\bar{\Pi}}$ is bipartite.⁸

Let \bar{R} be the restriction of R to $\bar{\Pi}$. We will show that for all $S \in \bar{\Pi}$, there exists $S' \in \bar{\Pi}$ such that $\overrightarrow{SS'}$ in $\vec{\Gamma}_{\bar{\Pi}}(\bar{R})$. Suppose there exists $S \in \bar{\Pi}$ such that for all $S' \in \bar{\Pi}$, $\overrightarrow{SS'}$ does not hold in $\vec{\Gamma}_{\bar{\Pi}}(\bar{R})$. Without loss of generality, let $S \in \bar{\Pi}_1$. Then $S \in \sigma_1$ and $S \notin \sigma_2$. Then, since σ_2 is a stable coalition structure for R , $S \notin \sigma_2$ implies, given *non-blocking* in Lemma 1, that there exists $\hat{S} \in \sigma_2$ such that $\overrightarrow{S\hat{S}}$ in $\vec{\Gamma}_{\Pi^*}(R)$. Since $\hat{S} \notin \bar{\Pi}$, either $\hat{S} \in (\sigma_1 \cap \sigma_2)$ or $\hat{S} \in \Pi^* \setminus (\sigma_1 \cup \sigma_2)$. If $\hat{S} \in (\sigma_1 \cap \sigma_2)$ then $S \in \sigma_1$ implies that σ_1 is not a feasible set. Since this is a contradiction, $\hat{S} \in \Pi^* \setminus (\sigma_1 \cup \sigma_2)$. Given that $\hat{S} \in \sigma_2$, this is not possible. Thus, we reached a contradiction.

Given that for all $S \in \bar{\Pi}$, there exists $S' \in \bar{\Pi}$ such that $\overrightarrow{SS'}$ in $\vec{\Gamma}_{\bar{\Pi}}(\bar{R})$, and given that the number of players is finite, it follows that $\vec{\Gamma}_{\bar{\Pi}}(\bar{R})$ contains at least one directed cycle \vec{C} . Since $\Gamma_{\bar{\Pi}}$ is bipartite, \vec{C} is an even directed cycle. Moreover, since $\bar{\Pi}_1$ and $\bar{\Pi}_2$ are feasible sets that partition $\bar{\Pi}$, and since $\bar{\Pi}_1 \subseteq \sigma_1$ and $\bar{\Pi}_2 \subseteq \sigma_2$, it follows that the two alternate

⁸An undirected graph $\Gamma = (V, E)$ is *bipartite* if there is a partition of the vertex set V into two sets V_1 and V_2 such that all edges in E connect pairs of vertices v_1 and v_2 such that $v_1 \in V_i$ and $v_2 \in V_j$ implies that $i \neq j$.

stable solutions of \vec{C} are in σ_1 and σ_2 , respectively. \square

A **top set** of a coalition formation problem $R \in \mathcal{R}^n(\Pi^*)$ is $\hat{\Pi} \subseteq \Pi^*$ such that $\hat{\Pi}$ is strongly connected, and for each pair S, S' such that $S \in \Pi^* \setminus \hat{\Pi}$ and $S' \in \hat{\Pi}$, $\overrightarrow{S S'}$ does not hold in $\vec{\Gamma}_{\Pi^*}(R)$.

Fix a stable coalition formation problem $R \in \mathcal{R}^n(\Pi^*)$. Then an **iterated top set $\hat{\Pi}$ of R** is a subset of Π^* such that there exist Π_1, \dots, Π_k for which

- (1) $k \geq 1$,
- (2) $\hat{\Pi} = \Pi_k$,
- (3) k is odd,
- (4) for all $t \in \{1, \dots, k\}$, $\Pi_t \subset \Pi_{IR}^*(R)$ such that $\Pi_t = \emptyset$ is possible for even t 's,
- (5) Π_1, \dots, Π_k are all disjoint,
- (6) Π_1 is a top set of $\vec{\Gamma}_{\Pi^*}(R)$,
- (7) for all odd $t \geq 3$, Π_t is a top set of $\vec{\Gamma}_{\Pi^* \setminus \bigcup_{i=1}^{t-1} \Pi_i}(R)$,
- (8) for all odd $t \geq 1$, if $\bar{\Pi}_t$ is a stable solution of Π_t , then for all even t ($t \geq 2$), $\Pi_t = \{S \in \Pi^* \setminus \bigcup_{i=1}^{t-1} \Pi_i : \text{there exists } S' \in \bar{\Pi}_{t-1} \text{ such that } \overrightarrow{S S'} \text{ in } \vec{\Gamma}_{\Pi^*}(R)\}$.

A **coalition formation problem $R \in \mathcal{R}^n(\Pi)$ is hierarchically stable** if it is stable and the following three conditions hold.

- (i) For each iterated top set of R each stable solution is in some stable coalition structure for R .
- (ii) For each even directed cycle that is disjoint from at least one odd directed cycle, odd directed anticycle, or directed pairwise cycle that is in an iterated top set of R , both alternate stable solutions are in a stable coalition structure for R .
- (iii) For each iterated top set of R that contains an odd directed cycle, odd directed anticycle, or directed pairwise cycle, there exists a stable solution for R which contains a maximal feasible set in each odd directed cycle, odd directed anticycle, and directed pairwise cycle

in this top set.

Lemma 4 *If a coalition formation problem $R \in \mathcal{R}^n(\Pi)$ is hierarchically stable then it is myopically stable.*

Part 1 of the Main Theorem: *If a coalition formation model $\Pi^* \subseteq \Pi$ is conditionally myopically stable then it does not have an inaccessible coalition configuration.*

Part 2 of the Main Theorem: *If a coalition formation model $\Pi^* \subseteq \Pi$ does not have an inaccessible coalition configuration then it is conditionally myopically stable.*

Step 1:

Fix a coalition formation model $\Pi^* \subseteq \Pi$ that does not have an inaccessible coalition configuration. Suppose, by contradiction, that Π^* is not conditionally myopically stable. Suppose there exist $\bar{\Pi} \subset \Pi$ and $\hat{\Pi} \subset \bar{\Pi}$ such that $\bar{\Pi}$ is conditionally myopically stable and $\hat{\Pi}$ is not conditionally myopically stable. Then, since $\hat{\Pi}$ is not conditionally myopically stable, there exists $\hat{R} \in \mathcal{R}^n(\hat{\Pi})$ such that $\hat{\Pi}$ is stable, but not myopically stable. But then, since $\hat{\Pi} \subset \bar{\Pi}$, $\hat{R} \in \mathcal{R}^n(\bar{\Pi})$. This means that $\bar{\Pi}$ is not conditionally myopically stable, which is a contradiction. Thus, if $\bar{\Pi} \subset \Pi$ is conditionally myopically stable, each subset of $\bar{\Pi}$ is also conditionally myopically stable.

Note, furthermore, that, e.g., for $\bar{\Pi} \subset \Pi^*$ such that $|\bar{\Pi}| = 2$, $\bar{\Pi}$ is conditionally myopically stable. Therefore, by eliminating one vertex at a time in Π^* , we will eventually find $\bar{\Pi} \subseteq \Pi^*$ such that it is not conditionally myopically stable, and $\bar{\Pi}$ is minimal in the sense that for all $\hat{\Pi} \subset \bar{\Pi}$, $\hat{\Pi}$ is conditionally myopically stable. Finally, since Π^* has no inaccessible coalition with respect to Π^* , $\bar{\Pi} \subset \Pi^*$ has no inaccessible coalition with respect to $\bar{\Pi}$. Therefore, we can assume, without loss of generality, that Π^* is minimal in the sense that for all $\hat{\Pi} \subset \Pi^*$, $\hat{\Pi}$ is conditionally myopically stable.

Since Π^* is not conditionally myopically stable, there exists $R \in \mathcal{R}^n(\Pi^*)$ such that R

is stable, but not myopically stable. Then $\Pi_{IR}^*(R) = \Pi^*$, since otherwise $\Pi_{IR}^*(R) \subset \Pi^*$, and $\Pi_{IR}^*(R)$ is not conditionally myopically stable, which contradicts the minimality of Π^* . Furthermore, there exists a connected component of $\vec{\Gamma}_{\Pi^*}(R)$ such that if we restrict R to this connected component, then this restriction is still stable, but not myopically stable. However, given the minimality of Π^* , this means that $\vec{\Gamma}_{\Pi^*}(R)$ is connected.

Step 2:

Since R is not myopically stable, Proposition 1 implies that R contains an odd directed cycle, an odd directed anticycle, or a pairwise directed cycle.

Since R is not myopically stable, Lemma 4 implies that R is not hierarchically stable. Since R is stable, this implies that one of conditions (i), (ii), or (iii) does not hold for R in the definition of a hierarchically stable coalition formation problem. If condition (ii) does not hold then the definition of a minimal inaccessible coalition configuration under Case 1 (“handcuffs”) is satisfied by $\vec{\Gamma}_{\Pi^*}(R)$. Since this is a contradiction, either condition (i) or (iii) must hold. We will consider these in turn.

Step 3: *Condition (i) does not hold for R .*

There is a top set of R such that some stable solution of the top set is not in any stable coalition structure for R .

Step 3a: Fix a top set $\hat{\Pi}$ of R such that some stable solution of $\hat{\Pi}$ for R is not in any stable coalition structure for R . Then there exists a sequence $\Pi_1, \Pi_2, \dots, \Pi_k$ such that $\Pi_k = \hat{\Pi}$ and $k \geq 1$ is odd, as described in the definition of a top set of R . We can assume, without loss of generality, that $\hat{\Pi} = \Pi_k$ is the first top set among the odd numbered Π_t ’s in this sequence which has a stable solution for R that is not in any stable coalition structure for R . Then, since all stable solutions of $\Pi_1, \Pi_3, \dots, \Pi_{k-2}$ for R are in a stable coalition structure for R , Lemma 1 implies that there exists a stable solution of Π_k for R such that it is in a stable coalition structure for R .

Let R^* be the restriction of R to $\Pi^* \setminus \bigcup_{t=1}^{k-1} \Pi_t$. Then R^* is stable, since there exists a stable solution of Π_k for R such that it is in a stable coalition structure for R . In particular, the part of the latter stable coalition structure which is restricted to $\Pi^* \setminus \bigcup_{t=1}^{k-1} \Pi_t$ is a stable coalition structure for R^* .

Next, we will show that R^* is not myopically stable. Let $\sigma \in \Sigma(\Pi^* \setminus \bigcup_{t=1}^{k-1} \Pi_t)$ be a coalition structure such that the stable solution of Π_k for R which is not in any stable coalition structure for R is contained in σ . Note that the latter stable solution of Π_k for R is not in any stable coalition structure for R^* either. Then, given that Π_k is a top set of $\vec{\Gamma}_{\Pi^* \setminus \bigcup_{t=1}^{k-1} \Pi_t}(R^*)$, it follows that R^* is not myopically stable.

In sum, R^* is stable and not myopically stable. Therefore, $\bar{\Pi} = \Pi^* \setminus \bigcup_{t=1}^{k-1} \Pi_t$ is not conditionally myopically stable. Then the minimality of Π^* implies that $\bigcup_{t=1}^{k-1} \Pi_t = \emptyset$, which means that $\hat{\Pi} = \Pi_k$ is a top set of $\vec{\Gamma}_{\Pi^*}(R)$. Thus, $k = 1$ and $\hat{\Pi} = \Pi_1$.

Step 3b: First note that Π_1 has at least two stable solutions for R . Then it follows from Lemma 3 that Π_1 contains an even directed cycle \vec{C} . Let \hat{C} be the set of coalitions in \vec{C} . Note that there are two stable solutions of \hat{C} for R such that one of them is not in any stable coalition structure for R , and the other one is in some stable coalition structure for R . Let \hat{R} be the restriction of R to $(\Pi^* \setminus \Pi_1) \cup \hat{C}$. Then a similar argument to the one presented in Step 3a for R^* implies that \hat{R} is stable and not myopically stable. Therefore, $\bar{\Pi} = (\Pi^* \setminus \Pi_1) \cup \hat{C}$ is not conditionally myopically stable. Then the minimality of Π^* implies that $\Pi_1 = \hat{C}$.

Step 3c: We will show that R restricted to $\Pi^* \setminus \Pi_1$ contains an odd directed cycle, an odd directed anticycle, or a pairwise directed cycle. Suppose not. Then R restricted to $\Pi^* \setminus \Pi_1$ is myopically stable, by Proposition 1. Note, furthermore, that this implies that for all subsets $\Pi' \subset \Pi^* \setminus \Pi_1$, R restricted to Π' contains no odd directed cycle, odd directed anticycle, or pairwise directed cycle, and thus R restricted to Π' is myopically stable as

well.

Note that since $\Pi_1 = \hat{C}$, R restricted to Π_1 is myopically stable. Then, given that the definition of a top set of R implies that for all $S \in \Pi_1$, and for all $\bar{S} \in \Pi^* \setminus \Pi_1$, we don't have $\overrightarrow{\bar{S}S}$ in $\overrightarrow{\Gamma}_{\Pi^*}(R)$, it follows that starting from an arbitrary coalition structure, we can find a myopic blocking sequence, by choosing blocking coalitions only in Π_1 , such that we reach a stable solution of Π_1 for R . Then, let $\Pi' \subseteq \Pi^* \setminus \Pi_1$ such that Π' contains each coalition in $\Pi^* \setminus \Pi_1$, except for the neighbors of the stable solution that we reached for Π_1 (which are not themselves in Π_1). Therefore, since R restricted to Π' is also myopically stable, we can find a myopic blocking sequence such that we reach a stable solution for Π' as well. This implies that R is myopically stable, which is a contradiction. Therefore, R restricted to $\Pi^* \setminus \Pi_1$ contains a an odd directed cycle, an odd directed anticycle, or a pairwise directed cycle.

Step 3d: We will show that for each coalition neighboring \hat{C} , the coalition is not a neighbor of any two coalitions in \hat{C} that are an odd distance away from each other. Suppose, by contradiction, that there exists $S' \notin \hat{C}$ such that S' is a neighbor of \hat{C} , and S' is a neighbor of two coalitions in \hat{C} that are an odd distance away from each other in \hat{C} . Let $\hat{C} = \{S_1, \dots, S_m\}$, where m is even, such that $\overrightarrow{S_m S_1}$, and for all $t \in \{1, \dots, m-1\}$, $\overrightarrow{S_t S_{t+1}}$ in $\overrightarrow{\Gamma}_{\Pi^*}(R)$. Since $\overrightarrow{\hat{C}}$ is an even directed cycle, the two stable solutions of \hat{C} for R are given by $\hat{C}_1 = \{S_1, S_3, \dots, S_{m-1}\}$ and $\hat{C}_2 = \{S_2, S_4, \dots, S_m\}$. Then, since S' is a neighbor of both \hat{C}_1 and \hat{C}_2 , there is no stable coalition structure for R that includes S' .

Let \bar{R} be the restriction of R to $\Pi^* \setminus \{S'\}$. Then \bar{R} is stable. Assume, without loss of generality, that \hat{C}_1 is in a stable coalition structure for R , and \hat{C}_2 is not in a stable coalition structure for R . Since \hat{C} is the only top set of $\overrightarrow{\Gamma}_{\Pi^* \setminus \{S'\}}(\bar{R})$, \hat{C}_1 is in a stable coalition structure for \bar{R} , and \hat{C}_2 is not in a stable coalition structure for \bar{R} . Therefore, \bar{R} is not myopically stable. In sum, \bar{R} is stable and not myopically stable. Thus, $\bar{\Pi} = \Pi^* \setminus \{S'\}$ is not conditionally myopically stable. This contradicts the minimality of Π^* . Therefore, for

each coalition neighboring \hat{C} , the coalition is not a neighbor of any two coalitions in \hat{C} that are an odd distance away from each other.

Step 3e: By Step 3b, $\Pi_1 = \hat{C}$ is an even cycle. By Step 3c, $\Pi^* \setminus \Pi_1$ contains an odd cycle, an odd anticycle, or a pairwise cycle. Fix such a C , C_- , or W . Clearly, \hat{C} and the fixed C , C_- , or W are disjoint. Since $\vec{\Gamma}_{\Pi^*}(R)$ is connected, Γ_{Π^*} is connected, and there exists a shortest path connecting \hat{C} and the fixed C , C_- , or W , via coalitions $\bar{S}_1, \dots, \bar{S}_k$ ($k \geq 2$), such that \bar{S}_1 is in \hat{C} , \bar{S}_k is in the fixed C , C_- , or W , \bar{S}_2 is the only neighbor of \hat{C} , and \bar{S}_{k-1} is the only neighbor of the fixed C , C_- , or W . Note, furthermore, that Step 3d implies that \bar{S}_2 is not a neighbor of any coalition in \hat{C} that is an odd distance away from \bar{S}_1 in \hat{C} . Now let $S = \bar{S}_1$ if k is even, and let S be a neighbor of \bar{S}_1 in \hat{C} if k is odd. Then the definition of a minimal inaccessible coalition configuration under Case 1 (“handcuffs”) is satisfied. Therefore, $\vec{\Gamma}_{\Pi^*}(R)$ has an inaccessible coalition configuration which satisfies the specifications in Case 1. This is a contradiction.

Therefore, condition (i) in the definition of a hierarchically stable coalition formation problem holds for R , which means that condition (iii) does not hold for R .

Step 4:

We will show that since condition (i) holds, by Step 3, it follows that the only top set of R is Π^* itself. Suppose, by contradiction, that Π^* is not a top set of R . Then there exists $\Pi_1 \subset \Pi^*$, such that it is a top set of R . Let \bar{R} be the restriction of R to Π_1 . Then, since R is stable and Π_1 is a top set of R , Π_1 has at least one stable solution for R . This means that \bar{R} is stable.

Case 1: \bar{R} is not myopically stable

Then Π_1 is not conditionally myopically stable. Thus, the minimality of Π^* implies that $\Pi^* = \Pi_1$, which is a contradiction, since $\Pi_1 \subset \Pi^*$.

Case 2: \bar{R} is myopically stable

Assume that there are $m \geq 1$ stable solutions $\sigma_1, \dots, \sigma_m$ of Π_1 for R . For all $t \in \{1, \dots, m\}$, let $\Pi'_t \subseteq \Pi^* \setminus \Pi_1$ such that Π'_t contains each coalition in $\Pi^* \setminus \Pi_1$, except for the neighbors of σ_t that are not in Π_1 . Suppose, by contradiction, that for all $t \in \{1, \dots, m\}$, R restricted to Π'_t is myopically stable. First note that, given the definition of a top set of R , for all $S \in \Pi_1$, and for all $\bar{S} \in \Pi^* \setminus \Pi_1$, we don't have $\overrightarrow{\bar{S}S}$ in $\overrightarrow{\Gamma}_{\Pi^*}(R)$. Then since \bar{R} is myopically stable, starting from an arbitrary coalition structure, we can find a myopic blocking sequence, by choosing blocking coalitions only in Π_1 , such that we reach a stable solution σ_t of Π_1 for R , where $t \in \{1, \dots, m\}$. Then, since R restricted to Π'_t is also myopically stable, we can find a myopic blocking sequence such that we reach a stable solution of Π'_t for R as well. This implies that R is myopically stable, which is a contradiction. Therefore, there exists $l \in \{1, \dots, m\}$ such that R restricted to Π'_l is not myopically stable.

Now note that since condition (i) holds in the definition of a hierarchically stable coalition formation problem, for all $t \in \{1, \dots, m\}$, the stable solution σ_t of Π_1 for R is in some stable coalition structure for R . This implies that for all $t \in \{1, \dots, m\}$, Π'_t has a stable solution for R . In particular, Π'_l has a stable solution for R . This means that R restricted to Π'_l is stable. Since R restricted to Π'_l is not myopically stable, this implies that Π'_l is not conditionally myopically stable. Since $\Pi'_l \subseteq \Pi^* \setminus \Pi_1$, and since $\Pi_1 \neq \emptyset$, given that Π_1 is a top set of R , we have $\Pi'_l \subset \Pi^*$. This contradicts the minimality of Π^* .

Therefore, Π^* itself is a top set of R .

Step 5: *Condition (iii) does not hold for R .*

If there is a top set of R such that it contains at least one odd directed cycle, odd directed anticycle, or pairwise directed cycle, then there is one such top set of R such that it has no stable solution for R which contains a maximal independent set in each odd directed cycle, odd directed anticycle, and pairwise directed cycle.

By Step 4, Π^* is a top set of R . Therefore, $\vec{\Gamma}_{\Pi^*}(R)$ is strongly connected. Since R is not myopically stable, Proposition 1 implies that R contains at least one odd directed cycle, odd directed anticycle, or pairwise directed cycle. Since condition (ii) in the definition of a hierarchically stable coalition formation problem does not hold for R , R has no stable solution which contains a maximal feasible set in each odd directed cycle, odd directed anticycle, and pairwise directed cycle.

Step 5a: By Proposition 1, since R is not myopically stable, there exists an odd directed cycle \vec{C} , and odd directed anticycle \vec{C}_- , or a pairwise directed cycle \vec{W} in $\vec{\Gamma}_{\Pi^*}(R)$. Then, given Step 4, and given that condition (ii) does not hold for R , there exists \vec{C} , \vec{C}_- or \vec{W} in $\vec{\Gamma}_{\Pi^*}(R)$ such that no stable coalition structure for R contains a maximal feasible set in the given \vec{C} , \vec{C}_- or \vec{W} . Fix such a \vec{C} , \vec{C}_- or \vec{W} .

Case A: \vec{C}

Let $C \subset \Pi_{IR}^*(R)$ be the set of coalitions in \vec{C} . Let \vec{C} be a directed k -cycle, where $k \geq 3$ is odd. Then, since a maximal feasible set in \vec{C} is of cardinality $\frac{k-1}{2}$, in any given stable coalition structure σ for R there are at most $\frac{k-3}{2}$ coalitions from C : $|\sigma \cap C| \leq \frac{k-3}{2}$. Furthermore, since \vec{C} is a directed cycle, each coalition $S \in C$ which is in a given stable coalition structure for R is such that there is another coalition $S' \in C$ with $\vec{S}S'$ in \vec{C} . Thus, there exist $S_1, S_2, S_3 \in C$ such that all pairs S_i, S_j with $i, j \in \{1, 2, 3\}$, $i \neq j$, are an odd distance away from each other in \vec{C} , and S_1, S_2, S_3 are not in any stable coalition structure for R . Let $\vec{\Pi}_{\vec{C}} = \{S_1, S_2, S_3\}$. Note that, since R is stable, *non-blocking* in Lemma 1 and the above argument imply that for all coalitions $S \in \vec{\Pi}_{\vec{C}}$, there exists $S' \in \Pi^* \setminus C$ such that $\vec{S}S'$ in $\vec{\Gamma}_{\Pi^*}(R)$.

Case B: \vec{C}_-

Let $C_- \subset \Pi_{IR}^*(R)$ be the set of coalitions in \vec{C}_- . Let \vec{C}_- be a directed k -anticycle, where $k \geq 7$ is odd. Then, since a maximal feasible set in \vec{C}_- is of cardinality 2, in any given

stable coalition structure σ for R there is at most one coalition from C_- : $|\sigma \cap C_-| \leq 1$. Furthermore, since $\overrightarrow{C_-}$ is a directed k -anticycle, where k is odd, if $\sigma \cap C_- = \{S\}$ then there are exactly $\frac{k-3}{2}$ coalitions $S' \in C_-$ such that $\overrightarrow{S'S}$ in $\overrightarrow{C_-}$. Thus, there exist $k - (\frac{k-3}{2} + 1) = \frac{k+1}{2}$ coalitions in C_- such that these $\frac{k+1}{2}$ coalitions form a path in the main cycle of $\overrightarrow{C_-}$, and they are not in any stable coalition structure for R . Let $\bar{\Pi}_{\overrightarrow{C_-}}$ be a set of such a $\frac{k+1}{2}$ coalitions in C_- . Note that, since R is stable, *non-blocking* in Lemma 1 and the above argument imply that for all coalitions $S \in \bar{\Pi}_{\overrightarrow{C_-}}$, there exists $S' \in \Pi^* \setminus C_-$ such that $\overrightarrow{SS'}$ in $\overrightarrow{\Gamma}_{\Pi^*}(R)$.

Case 3: \overrightarrow{W}

Let $W \subset \Pi_{IR}^*(R)$ be the set of coalitions in \overrightarrow{W} . Since a maximal feasible set in \overrightarrow{W} is of cardinality one, no coalition from W is in any stable coalition structure for R . Therefore, since R is stable, *non-blocking* in Lemma 1 implies that for all coalitions $S \in W$, there exists $S' \in \Pi^* \setminus W$ such that $\overrightarrow{SS'}$ in $\overrightarrow{\Gamma}_{\Pi^*}(R)$.

For the rest of this proof, let \overrightarrow{G} denote the fixed \overrightarrow{C} , $\overrightarrow{C_-}$, or W , and let $\bar{\Pi}_{\overrightarrow{G}} = \bar{\Pi}_{\overrightarrow{C}}$, $\bar{\Pi}_{\overrightarrow{C_-}}$, or W , accordingly. Moreover, let G denote the set of coalitions in \overrightarrow{G} . Note that the above implies that for all stable coalition structures σ for R , $\sigma \cap \bar{\Pi}_{\overrightarrow{G}} = \emptyset$. Furthermore, $\bar{\Pi}_{\overrightarrow{G}} \in \Omega_G$, where Ω_G is as defined for the Main Theorem in Section 5.

Step 5b: We will show that for all $S \in \Pi^*$ such that there exists a stable coalition structure $\sigma \in \Sigma(\Pi^*)$ with $S \notin \sigma$, the following holds. Since R is not myopically stable, we can fix a coalition structure $\bar{\sigma} \in \Sigma(\Pi^*)$ such that $\bar{\sigma}$ does not converge to a stable coalition structure for R . Then either $S \in \bar{\sigma}$ or there exists $\hat{\sigma} \in \Sigma(\Pi^*)$ such that $\bar{\sigma}$ converges to $\hat{\sigma}$ for R , and $S \in \hat{\sigma}$.

Fix $S \in \Pi^*$ such that there exists a stable coalition structure $\sigma \in \Sigma(\Pi^*)$ with $S \notin \sigma$. Let R^* be the restriction of R to $\Pi^* \setminus \{S\}$. Note that σ is a stable coalition structure for R^* , and thus R^* is stable. Furthermore, by the minimality of Π^* , $\Pi^* \setminus \{S\}$ is conditionally myopically stable. Therefore, since R^* is stable, it is myopically stable.

Given that R is not myopically stable, we can fix a coalition structure $\bar{\sigma} \in \Sigma(\Pi^*)$ such that $\bar{\sigma}$ does not converge to a stable coalition structure for R . Suppose, by contradiction, that $S \notin \bar{\sigma}$ and there is no $\hat{\sigma} \in \Sigma(\Pi^*)$ such that $S \in \hat{\sigma}$ and $\bar{\sigma}$ converges to $\hat{\sigma}$ for R . Note that *non-blocking* in Lemma 1 implies that a blocking coalition S' of a coalition structure $\tilde{\sigma} \in \Sigma(\Pi^*)$ for R , where $S' \neq S$ and $S \notin \tilde{\sigma}$, is also a blocking coalition of $\tilde{\sigma}$ for R^* and $\tilde{\sigma} \in \Sigma(\Pi^*)$. Note that $\bar{\sigma} \in \Sigma(\Pi^*)$. Then $\bar{\sigma}$ does not converge to a stable coalition structure for R^* . Since R^* is myopically stable, this is a contradiction.

Step 5c: We will show that R has a unique stable coalition structure.

Suppose R does not have a unique stable coalition structure. Then, since R is stable, R has at least two stable coalition structures. Thus, Lemma 3 implies that there exists an even directed cycle \vec{C} such that its two alternate solutions are in two different stable coalition structures for R . Let \hat{C} be the set of coalitions in \vec{C} . Then, for each coalition $S \in \hat{C}$, there exists a stable coalition structure $\sigma \in \Sigma(\Pi^*)$ such that $S \notin \sigma$. Then Step 5b implies that for each coalition $S \in \hat{C}$ and for all coalition structures $\bar{\sigma} \in \Sigma(\Pi^*)$ such that $\bar{\sigma}$ does not converge to a stable coalition structure for R , either $S \in \bar{\sigma}$ or there exists $\hat{\sigma} \in \Sigma(\Pi^*)$ such that $\bar{\sigma}$ converges to $\hat{\sigma}$ for R and $S \in \hat{\sigma}$. Then *feasibility* in Lemma 1 implies that for all $S \in \hat{C}$ and for all coalition structures $\bar{\sigma} \in \Sigma(\Pi^*)$ such that $\bar{\sigma}$ does not converge to a stable coalition structure for R , there exists $\hat{\sigma} \in \Sigma(\Pi^*)$ such that $\bar{\sigma}$ converges to $\hat{\sigma}$ for R and S is a blocking coalition of $\hat{\sigma}$ for R . Let R^* be the restriction of R to \hat{C} . Since $\vec{\Gamma}_{\hat{C}}(R^*) = \vec{C}$ is an even directed cycle, R^* is stable. Then, however, we can derive a contradiction using the argument in the proof of Proposition 1. Therefore, R has a unique stable coalition structure.

Step 5d: We will show that for all $\bar{\sigma} \in \Sigma(\Pi^*)$ such that $\bar{\sigma}$ does not converge to a stable coalition structure for R and for all $\hat{\sigma} \in \Sigma(\Pi^*)$ such that $\hat{\sigma}$ is the S -successor of $\bar{\sigma}$, $S \in G$.

Suppose there exists $S'' \in G$ such that for all $\bar{\sigma} \in \Sigma(\Pi^*)$ that does not converge to a

stable coalition structure for R , S'' is not a blocking coalition of $\bar{\sigma}$ for R . We will show that this is a contradiction.

By Step 5c, there exists $\sigma \in \Sigma(\Pi^*)$ that is the unique stable coalition structure for R . Fix $\sigma_1 \in \Sigma(\Pi^*)$ such that σ_1 does not converge to a stable coalition structure for R . Let \vec{G} be an odd directed k -cycle with $k \geq 3$, or an odd directed k -anticycle with $k \geq 7$, or a directed pairwise k -cycle with $k \geq 4$. Note that *feasibility* in Lemma 1 implies that $|\sigma \cap G| \leq k - 1$. Therefore, there exist two neighbors S and S' in \vec{G} such that $\overrightarrow{S S'}$ in \vec{G} and $S, S' \notin \sigma$. In particular, we can choose S and S' such that $\overrightarrow{S S'}$ in the main cycle of \vec{G} . *Feasibility* in Lemma 1 implies that at most one of S and S' is in σ_1 . Let $S \notin \sigma_1$, without loss of generality. Then Step 5b implies that there exists $\sigma_2 \in \Sigma(\Pi^*)$ such that σ_1 converges to σ_2 for R and $S \in \sigma_2$. Note that σ_2 does not converge to a stable coalition structure for R . Then, given that S and S' are neighbors in \vec{G} , *feasibility* in Lemma 1 implies that $S' \notin \sigma_2$. Continuing this argument iteratively, we can find an infinite sequence of coalition structures $(\sigma_t)_{t=2,3,\dots}$ such that, for all $t \geq 1$, $\sigma_t \in \Sigma(\Pi^*)$, σ_t converges to σ_{t+1} for R , and, for all $t \geq 2$, if t is even then $S \in \sigma_t$ and $S' \notin \sigma_t$, and if t is odd then $S' \in \sigma_t$ and $S \notin \sigma_t$.

Note that $S' \in \sigma_3$ and $S \in \sigma_4$, where σ_3 converges to σ_4 for R . Then, $\overrightarrow{S S'}$ in \vec{G} implies that there exist $\bar{\sigma} \in \Sigma(\Pi^*)$ and $\bar{S} \in \Pi^*$ such that σ_3 converges to $\bar{\sigma}$ for R , $\bar{\sigma}$ converges to σ_4 for R , $\bar{S} \neq S, S'$, $\overrightarrow{\bar{S} \bar{S}}$ in $\vec{\Gamma}_{\Pi^*}(R)$, and $\bar{S} \in \bar{\sigma}$. Suppose $\bar{S} \notin G$. We will show that this is a contradiction.

Since $\bar{\sigma}$ converges to σ_4 for R and σ_4 converges to σ_5 for R , $\bar{\sigma}$ converges to σ_5 for R . Then, given that $\overrightarrow{S' \bar{S}}$ in $\vec{\Gamma}_{\Pi^*}(R)$, $\bar{S} \in \bar{\sigma}$ and $S' \in \sigma_5$, there exist $\tilde{\sigma} \in \Sigma(\Pi^*)$ and $\tilde{S} \in \Pi^*$ such that $\bar{\sigma}$ converges to $\tilde{\sigma}$ for R , $\tilde{\sigma}$ converges to σ_5 for R , $\tilde{S} \neq S', \tilde{S} \neq \bar{S}$, $\overrightarrow{\tilde{S} \tilde{S}}$ in $\vec{\Gamma}_{\Pi^*}(R)$, and $\tilde{S} \in \tilde{\sigma}$. Given that there is a finite number of players, if we continue this argument iteratively we will find a general cycle \vec{C}^* in $\vec{\Gamma}_{\Pi^*}(R)$ and an infinite sequence of coalition structures $(\sigma'_t)_{t=1,2,\dots}$ such that for all $t \geq 1$, σ'_t does not converge to a stable coalition structure for R , σ'_{t+1} is a successor of σ'_t for R , and for all $S^* \in C^*$, there exists an infinite

number of $t \geq 1$ such that σ_{t+1} is the S^* -successor of σ_t . Let R^* be the restriction of R to C^* . For all $t \geq 1$, let $\sigma_t^* = \sigma_t \cap C^*$.

Renumber σ_t^* 's as in Step 1 of the proof of Proposition 1. This results in $(\sigma_t^*)_{t=1^*, 2^*, \dots}$ such that for all $t = 1^*, 2^*, \dots$, σ_{t+1}^* is a successor of σ_t^* , and for all $t = 1^*, 2^*, \dots$, there exists $S^* \in C^*$ such that σ_{t+1}^* is the S^* -successor of σ_t^* for R^* . Then, if R^* is stable we can derive a contradiction using the argument of the proof of Proposition 1. Therefore, R^* is not stable.

Then Pápai (2002) implies that $\vec{\Gamma}_{C^*}(R^*)$ contains an odd directed cycle, odd directed anticycle, or directed pairwise cycle. Let \hat{C} denote the set of coalitions in such an odd directed cycle, odd directed anticycle, or directed pairwise cycle. Note that either $\hat{C} = C^*$, given that $\vec{\Gamma}_{\Pi^*}(R^*) = \vec{C}^*$ is a general cycle, or $\hat{C} \subset C^*$. If $\hat{C} \subset C^*$, \hat{C} satisfies all the properties that C^* satisfies, except that $S, S' \in C^* \setminus \hat{C}$ is possible. Suppose, by contradiction, that $S \in C^* \setminus \hat{C}$. Since $S \notin \sigma$, R restricted to $\Pi^* \setminus \{S\}$ is stable. Since $S \notin \hat{C}$, $\hat{C} \subset \Pi^* \setminus \{S\}$, which implies that R restricted to $\Pi^* \setminus \{S\}$ is not myopically stable, as shown above. This contradicts the minimality of Π^* . Thus, $S \in \hat{C}$. Since $S' \notin \sigma$, we can show similarly that $S' \in \hat{C}$. Therefore, we can assume without loss of generality that $\hat{C} = C^*$.

Step 5b implies that, for all $t = 1^*, 2^*, \dots$, and for all $\hat{S} \in \Pi^* \setminus C^*$ such that $\hat{S} \notin \sigma$, $\hat{S} \in \sigma_t^*$. Therefore, $\Pi^* \setminus C^*$ can be partitioned into $\sigma \cap (\Pi^* \setminus C^*)$ and $\sigma_t^* \cap (\Pi^* \setminus C^*)$. Given *feasibility* in Lemma 1, this means that $\vec{\Gamma}_{\Pi^* \setminus C^*}$ is bipartite. Since R is stable, $\Pi^* \setminus C^* \neq \emptyset$.

Suppose that for all $\tilde{S} \in \Pi^*$ such that $\tilde{S} \in \sigma \setminus C^*$, there exists $\check{S} \in \Pi^* \setminus C^*$ such that $\vec{\tilde{S}\check{S}}$ in $\vec{\Gamma}_{\Pi^*}(R)$. Then, for all $t = 1^*, 2^*, \dots$, $\check{S} \in \sigma_t^*$, and thus $\check{S} \in \sigma_t^* \setminus C^*$. Thus, given that $\vec{\Gamma}_{\Pi^* \setminus C^*}$ is bipartite, there exists an even directed cycle in $\Pi^* \setminus C^*$. Note, however, that since C^* is an odd cycle, odd anticycle, or pairwise cycle, we can use the same arguments as in Steps 3e and 3d to get a contradiction. Therefore, there exists $\tilde{S} \in \Pi^*$ such that $\tilde{S} \in \sigma \setminus C^*$ and there is no $\check{S} \in \Pi^* \setminus C^*$ such that $\vec{\tilde{S}\check{S}}$ in $\vec{\Gamma}_{\Pi^*}(R)$. Since $\vec{\Gamma}_{\Pi^*}(R)$ is strongly connected, by Step 4, this means that for all $\check{S} \in \Pi^*$ such that $\vec{\tilde{S}\check{S}}$ in $\vec{\Gamma}_{\Pi^*}(R)$, $\check{S} \in C^*$.

Let $\check{\Pi} = \{\check{S} \in \Pi^* : \overrightarrow{\check{S}\check{S}} \text{ in } \overrightarrow{\Gamma}_{\Pi^*}(R)\}$. Since for all $t = 1^*, 2^*, \dots$, $\check{S} \notin \sigma_t^*$, there exists $\check{S} \subseteq \check{\Pi}$ such that $\check{S} \in \Omega_{C^*}$ (where Ω_{C^*} is defined similarly to Ω_G in Section 5). Furthermore, since $\check{S} \in \sigma$, by *feasibility* in Lemma 1, for all $\check{S} \in \check{\Pi}$, $\check{S} \notin \sigma$.

Then, given that R is stable, and given that C^* is an odd directed cycle, odd directed anticycle, or directed pairwise cycle, Lemma 1 implies that there exists $\check{\check{\Pi}} \subseteq \check{\Pi}$ such that $\check{\check{\Pi}} \in \Omega_{C^*}$ and for all $\check{S} \in \check{\check{\Pi}}$, there exists $\bar{S} \in \sigma \setminus C^*$. Note that since $\Pi^* \setminus C^*$ is bipartite and $\overrightarrow{\Gamma}_{\Pi^*}(R)$ is strongly connected, by Step 4, this means that conditions (1) - (3) are satisfied in the definition of a minimal inaccessible coalition configuration under Case 2 (“cages”). Note also that, since R is stable, conditions (A) and (B) in the same definition also hold. Therefore, $\overrightarrow{\Gamma}_{\Pi^*}(R)$ has an inaccessible coalition configuration which satisfies the specifications in Case 2 (“cages”). This is a contradiction.

Therefore, for all $S'' \in G$, there exists $\bar{\sigma} \in \Sigma(\Pi^*)$ that does not converge to a stable coalition structure for R , such that S'' is a blocking coalition of $\bar{\sigma}$ for R . Then we can construct C^* as before, such that $C^* = G$, and C^* has all the properties found above. Therefore, for all $\bar{\sigma} \in \Sigma(\Pi^*)$ such that $\bar{\sigma}$ does not converge to a stable coalition structure for R and for all $\hat{\sigma} \in \Sigma(\Pi^*)$ such that $\hat{\sigma}$ is the S'' -successor of $\bar{\sigma}$, $S'' \in G$.

Step 5e: Let $\hat{\Pi}_{\overrightarrow{G}} = \{S \in G : \text{there exists } S' \in \Pi^* \setminus G \text{ such that } \overrightarrow{S S'} \text{ in } \overrightarrow{\Gamma}_{\Pi^*}(R)\}$. We will show that for all $S \in \hat{\Pi}_{\overrightarrow{G}}$, there exists a stable coalition structure $\sigma \in \Sigma(\Pi^*)$ for R such that $S \notin \sigma$.

Note first that Step 5a implies that $\bar{\Pi}_{\overrightarrow{G}} \subseteq \hat{\Pi}_{\overrightarrow{G}}$. Suppose there exists $S \in \hat{\Pi}_{\overrightarrow{G}}$ such that for all stable coalition structures $\sigma \in \Sigma(\Pi^*)$ for R , $S \in \sigma$. Let $S' \in \Pi^* \setminus G$ such that $\overrightarrow{S S'}$ in $\overrightarrow{\Gamma}_{\Pi^*}(R)$. Then, by *feasibility* in Lemma 1, $S' \notin \sigma$ for any stable coalition structure σ for R .

Fix $\bar{\sigma} \in \Sigma(\Pi^*)$ such that $\bar{\sigma}$ does not converge to a stable coalition structure for R . Then, since $S' \notin \sigma$ for any stable coalition structure σ for R , Step 5b implies that either $S' \in \bar{\sigma}$

or there exists $\hat{\sigma} \in \Sigma(\Pi^*)$ such that $\bar{\sigma}$ converges to $\hat{\sigma}$ for R and $S' \in \hat{\sigma}$. The latter would mean that $\bar{\sigma}$ converges to a coalition structure for R which has S' as a blocking coalition. Since $S' \notin G$, this contradicts Step 5d. Therefore, $S' \in \bar{\sigma}$. Furthermore, note that Step 5b implies that for all $\tilde{\sigma} \in \Sigma(\Pi^*)$ such that $\bar{\sigma}$ converges to $\tilde{\sigma}$ for R , $S' \in \tilde{\sigma}$. Since $\overrightarrow{S S'}$ in $\overrightarrow{\Gamma}_{\Pi^*}(R)$, this means, however, that for all $\tilde{\sigma} \in \Sigma(\Pi^*)$ such that $\bar{\sigma}$ converges to $\tilde{\sigma}$ for R , S is not a blocking coalition of $\tilde{\sigma}$ for R . Since $S \in G$, this contradicts Step 5d. Therefore, for all $S \in \hat{\Pi}_{\overrightarrow{G}}$, there exists a stable coalition structure $\sigma \in \Sigma(\Pi^*)$ for R such that $S \in \sigma$.

Step 5f: Fix $\bar{\sigma} \in \Sigma(\Pi^*)$ such that $\bar{\sigma}$ does not converge to a stable coalition structure for R . We will show that $\bar{\sigma} \cap \hat{\Pi}_{\overrightarrow{G}} \neq \emptyset$.

Suppose, by contradiction, that $\bar{\sigma} \cap \hat{\Pi}_{\overrightarrow{G}} = \emptyset$. Let $\sigma \in \Sigma(\Pi^*)$ be a stable coalition structure for R . Then, by Step 5c, σ is the unique stable coalition structure for R , and thus Step 5e implies that $\sigma \cap \hat{\Pi}_{\overrightarrow{G}} = \emptyset$.

Let \bar{R} be the restriction of R to $\Pi^* \setminus \hat{\Pi}_{\overrightarrow{G}}$. Then, since $\sigma \cap \hat{\Pi}_{\overrightarrow{G}} = \emptyset$, σ is a stable coalition structure for \bar{R} . Suppose σ is not the only stable coalition structure for \bar{R} . Then Lemma 3 implies that there exists an even directed cycle \overrightarrow{D} in $\Pi^* \setminus \hat{\Pi}_{\overrightarrow{G}}$ such that the two alternative solutions of \overrightarrow{D} are in two different stable coalition structures for \bar{R} . Note that \overrightarrow{D} cannot be contained in $\Pi^* \setminus G$, because otherwise, given that G is an odd cycle, odd anticyle, or pairwise cycle, we can use the same arguments as in Steps 3e and 3d to derive a contradiction. Therefore, there exists $S \in G \setminus \hat{\Pi}_{\overrightarrow{G}}$ such that S is in \overrightarrow{D} . Since for all $S \in G \setminus \hat{\Pi}_{\overrightarrow{G}}$, there is no $S' \in \Pi^* \setminus G$ such that $\overrightarrow{S S'}$ in $\overrightarrow{\Gamma}_{\Pi^*}(R)$, and since \overrightarrow{D} is in $\Pi^* \setminus \hat{\Pi}_{\overrightarrow{G}}$, this means that \overrightarrow{D} is contained in $G \setminus \hat{\Pi}_{\overrightarrow{G}}$. Since $\bar{\Pi}_{\overrightarrow{G}} \subseteq \hat{\Pi}_{\overrightarrow{G}}$, where $\bar{\Pi}_{\overrightarrow{G}} \in \Omega_G$, this means that \overrightarrow{G} is a directed k -anticycle $\overrightarrow{C_-}$, where $k \geq 7$ is odd. Note, however, that $C_- \setminus \bar{\Pi}_{\overrightarrow{C_-}}$ consists of $\frac{k-1}{2}$ coalitions, such that these coalitions form a path in the main cycle of $\overrightarrow{C_-}$. Therefore, $\overrightarrow{\Gamma}_{C_- \setminus \bar{\Pi}_{\overrightarrow{C_-}}}$ does not contain a directed cycle. Since \overrightarrow{D} is contained in $G \setminus \hat{\Pi}_{\overrightarrow{C_-}}$, this is a contradiction. Thus, σ is the unique stable coalition structure for \bar{R} .

Since \bar{R} is stable and $\bar{\Pi}_{\bar{G}} \neq \emptyset$, the minimality of Π^* implies that \bar{R} is myopically stable. Note that since $\hat{\sigma} \cap \hat{\Pi}_{\bar{G}} = \emptyset$, $\hat{\sigma} \in \Sigma(\Pi^* \setminus \hat{\Pi}_{\bar{G}})$. Therefore, since \bar{R} is myopically stable, $\hat{\sigma}$ converges to the unique stable coalition structure σ for \bar{R} . Now note that if S is a blocking coalition of some coalition structure $\tilde{\sigma} \in \Sigma(\Pi^* \setminus \hat{\Pi}_{\bar{G}})$ for \bar{R} , then $\tilde{\sigma} \in \Sigma(\Pi^*)$ and S is also a blocking coalition of $\tilde{\sigma}$ for R . This means that $\hat{\sigma}$ converges to σ for R . Since σ does not converge to a stable coalition structure for R , this is a contradiction. Therefore, $\bar{\sigma} \cap \hat{\Pi}_{\bar{G}} \neq \emptyset$.

Step 5g: We will show that $\vec{\Gamma}_{\Pi^*}(R)$ has an inaccessible coalition configuration which satisfies the specifications in Case 2 (“cages”).

By Step 5c, there exists $\sigma \in \Sigma(\Pi^*)$ such that σ is the unique stable coalition structure for R . Then Steps 5b and 5d imply that for all $\bar{\sigma} \in \Sigma(\Pi^*)$ such that $\bar{\sigma}$ does not converge to a stable coalition structure for R , and for all $S \in \Pi^* \setminus G$ such that $S \notin \sigma_1$, $S \in \bar{\sigma}$. Note that this and *feasibility* in Lemma 1 imply that $\Pi^* \setminus G$ is bipartite. Then, for all $S, S' \in \Pi^* \setminus G$ such that S and S' are neighbors in Γ_{Π^*} , if $S \in \sigma$, then $S' \notin \bar{\sigma}$.

Fix $S \in \hat{\Pi}_{\bar{G}}$. Then, since $\vec{\Gamma}_{\Pi^*}(R)$ is strongly connected, by Step 4, there exists a directed path $\{(S_1, S_2), \dots, (S_{k-1}, S_k)\}$ in $\vec{\Gamma}_{\Pi^*}(R)$ such that $S_1 = S$, $k \geq 2$, for all $t \in \{2, \dots, k\}$, $S_t \in \Pi^* \setminus G$, it is the shortest path from S_2 to S_k , and there is no $S' \in \Pi^* \setminus G$ such that $\overrightarrow{S_k S'}$ in $\vec{\Gamma}_{\Pi^*}(R)$. Given that $\Pi^* \setminus G$ is bipartite, for all $t \in \{1, \dots, k\}$, if t is even then $S_t \in \sigma$, and if t is odd then $S_t \in \bar{\sigma}$, for all $\bar{\sigma} \in \Sigma(\Pi^*)$ that do not converge to a stable coalition structure for R .

Let $\tilde{\Pi} = \{S' \in \Pi^* : \overrightarrow{S_k S'} \text{ in } \vec{\Gamma}_{\Pi^*}(R)\}$. Since, for all $S' \in \Pi^* \setminus G$, $S' \notin \tilde{\Pi}$, strong connectedness of $\vec{\Gamma}_{\Pi^*}(R)$, by Step 4, implies that there exists $S' \in G$ such that $S' \in \tilde{\Pi}$. In sum, $\tilde{\Pi} \subseteq G$ and $\tilde{\Pi} \neq \emptyset$.

Given that S_k is a neighbor of G , Step 5d and *feasibility* in Lemma 1 imply that $S_k \notin \bar{\sigma}$. Then $S_k \in \sigma$ and k is even. Then, since $\Pi^* \setminus G$ is bipartite, and given *feasibility* in Lemma 1,

for all $t, t' \in \{2, \dots, k\}$ such that $t < t'$ and $t' - t$ is even, neither $\overrightarrow{S_t S_{t'}}$ nor $\overrightarrow{S_{t'} S_t}$ holds. Furthermore, fix $t, t' \in \{2, \dots, k\}$ such that $t < t'$, $t + 1 \neq t'$, and $t' - t$ is odd. Then $\overrightarrow{S_t S_{t'}}$ does not hold since $\{(S_2, S_3), \dots, (S_{k-1}, S_k)\}$ is a shortest path. Note, moreover, that if $\overrightarrow{S_t S_{t'}}$ holds then $S_t, \dots, S_{t'}$ form an even directed cycle in $\Pi^* \setminus G$. Then Steps 3e and 3d can be applied to get a contradiction. Therefore, for all $t, t' \in \{2, \dots, k\}$ such that $t < t'$ and $t + 1 = t'$, neither $\overrightarrow{S_t S_{t'}}$ nor $\overrightarrow{S_{t'} S_t}$ holds. This means that S_1 and S_k are in a cycle in $\Gamma_{\hat{\Pi}}$. Furthermore, since k is even, the path from S_1 to S_k is odd.

Given that $\tilde{\Pi} \subseteq G$, $\tilde{\Pi} \neq \emptyset$, and given that $S_k \notin \bar{\sigma}$ for all $\bar{\sigma} \in \Sigma(\Pi^*)$ such that $\bar{\sigma}$ does not converge to a stable coalition structure for R , for all such $\bar{\sigma}$, $\bar{\sigma} \cap \tilde{\Pi} \neq \emptyset$. Let the cardinality of a maximal feasible set of $G \setminus \hat{\Pi}_{\overrightarrow{G}}$ be denoted by l . Then Step 5d and 5f imply that, for all $\bar{\sigma} \in \Sigma(\Pi^*)$ such that $\bar{\sigma}$ does not converge to a stable coalition structure for R , $|\bar{\sigma} \cap G| \geq l + 1$. Therefore, there exists $\check{\Pi} \subseteq \tilde{\Pi}$ such that $\check{\Pi} \in \Omega_G$. Note that this holds for all $S \in \hat{\Pi}_{\overrightarrow{G}}$. Since R is stable, conditions (A) and (B) in the definition of a minimal inaccessible coalition configuration under Case 2 (“cages”) are satisfied. Thus, $\overrightarrow{\Gamma}_{\Pi^*}(R)$ has an inaccessible coalition configuration which satisfies the specifications in Case 2 (“cages”). This is a contradiction.

Therefore, condition (iii) in the definition of a hierarchically stable coalition formation problem holds for R . This is a contradiction, and the proof is completed.

References

- Banerjee, S., H. Konishi, T. Sönmez, (2001), “Core in a Simple Coalition Formation Game,” *Social Choice and Welfare* 18: 135-153.
- Bogomolnaia, A., M. Jackson (2002), “The Stability of Hedonic Coalition Structures,” *Games and Economic Behavior*, 38: 201-230.
- Cechlárová, K., A. Romero-Medina (2001) “Stability in Coalition Formation Games,” *International Journal of Game Theory*, 29: 487-494.
- Diamantoudi, E., E. Miyagawa, L. Xue (2002), “Random Paths to Stability in the Roommate Problem,” mimeo.
- Chung, K.-S. (2000), “On the Existence of Stable Roommate Matchings,” *Games and Economic Behavior* 33: 206-230.
- Drèze, J., J. Greenberg (1980), “Hedonic Coalitions: Optimality and Stability,” *Econometrica* 48: 987-1003.
- Feldman, A. (1974), “Recontracting Stability,” *Econometrica* 42: 35-44.
- Gale, D., L. Shapley (1962), “College Admissions and the Stability of Marriage,” *American Mathematical Monthly* 69: 9-15.
- Green, J. (1974), “The Stability of Edgeworth’s Recontracting Process,” *Econometrica* 42: 21-34.
- Jackson, M., A. Watts (2002), “The Evolution of Social and Economic Networks,” *Journal of Economic Theory* 106: 265-295.
- Ma, J. (1996), “On Randomized Matching Mechanisms,” *Economic Theory* 8: 377-381.

Pápai, S. (2002), “The Existence of Stable Hedonic Coalitions,” mimeo.

Pápai, S. (2003), “Unique Stability in Simple Coalition Formation Games,” mimeo.

Roth, A., M. Sotomayor (1990), *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*, Econometric Society Monograph 18, Cambridge: Cambridge University Press.

Roth, A., J. Vande Vate (1990), “Random Paths to Stability in Two-Sided Matching,” *Econometrica* 58: 1475-1480.

Figure 1: Examples of minimal inaccessible coalition configurations
 Case 1: "Handcuffs"

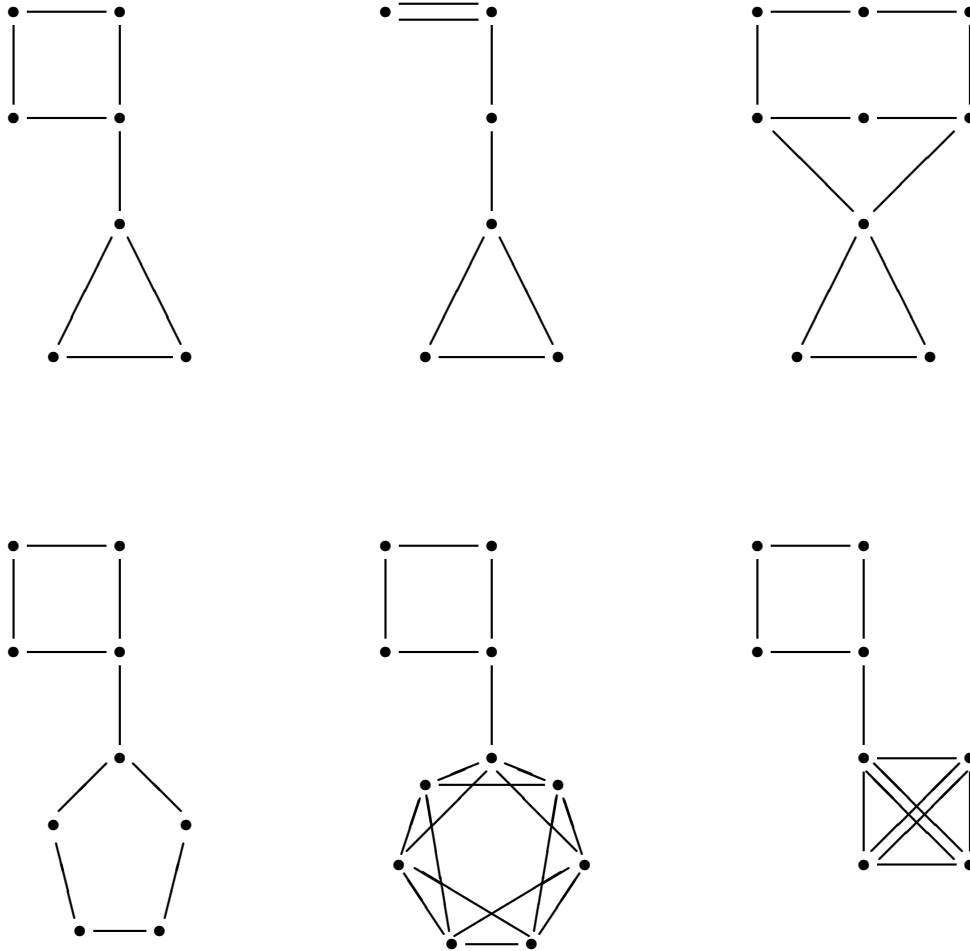


Figure 2: Examples of minimal inaccessible coalition configurations
 Case 2: "Cages"

