On the Stability of the Two-sector Neoclassical Growth Model with Externalities*

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Abstract

This paper explores the local stability properties of the steady state in the two-sector neoclassical growth model with sector–specific externalities. We show analytically that capital adjustment costs of *any* size preclude local indeterminacy nearby the steady state for every empirically plausible specification of the model parameters. More specifically, we show that when capital adjustment costs of *any* size are considered, a necessary condition for local indeterminacy is an upward-sloping labor demand curve in the capital-producing sector, which in turn requires an implausibly strong externality. We show numerically that capital adjustment costs of plausible size imply determinacy nearby the steady state for empirically plausible specifications of the other model parameters. These findings contrast sharply with the previous finding that local indeterminacy occurs in the two-sector model for a wide range of plausible parameter values when capital adjustment costs are abstracted from.

Keywords: capital adjustment costs; determinacy; externality; local indeterminacy; stability.

JEL classification: E0; E3.

1 Introduction

In this paper, we study the local stability of the steady state of the neoclassical growth model. Local stability analysis provides important information about the local uniqueness of equilibrium close to the steady state and about the type of business cycles that can occur in the model economy. In particular, if the steady state is saddle–path stable, then all nearby equilibria are locally unique, or determinate. With determinacy, business cycles require shocks to total factor productivity and they are typically efficient. It has been argued that this type of business cycle should not be stabilized. In contrast, if the steady state is stable, then a continuum of nearby equilibrium paths converge to the steady state implying a severe form of local non-uniqueness of equilibrium that is called local indeterminacy. With local indeterminacy, business cycles can originate from self-fulfilling shocks to individual beliefs and they can be inefficient. It has been argued that this second type of business cycle should be stabilized. Since both determinacy and local indeterminacy are theoretically possible, we ask which of them prevails for empirically plausible choices of the parameter values.

We focus our attention on a class of two-sector neoclassical growth models with sector–specific positive externalities, in which one sector produces a consumption good and the other sector produces the capital goods for both sectors. This class of models has been the focus of recent research on self-fulfilling business cycles; see e.g. Benhabib and Farmer (1996), Perli (1998), Weder (1998), Schmitt-Grohe (2000), and Harrison and Weder (2001). The reason is that local indeterminacy can occur for mild, empirically plausible externalities in the capital–producing sector, which are consistent with downward sloping labor demand curve. In contrast, in the class of standard one-sector neoclassical growth models, local indeterminacy requires the strengths of the externalities to be higher than is empirically plausible; see e.g. Benhabib and Farmer (1994) and Farmer and Guo (1994). In fact it requires such strong externalities that the labor demand curve becomes upward sloping, which leads to awkward economic implications

[Aiyagari (1995)].[1]

Our main finding is that the occurrence of local indeterminacy in the two–sector neoclassical growth model depends critically on the shape of the production possibility frontier between the two new capital goods. Specifically, we show two results. Our first result is analytical: we find that if the production possibility frontier between the two capital goods is strictly concave (meaning that the two capital goods are imperfect substitutes), then local indeterminacy does not occur for degrees of increasing returns that are consistent with downward sloping labor demand curve. This is in sharp contrast to the model with a linear production possibility frontier (meaning that the two capital goods are perfect substitutes) where local indeterminacy can occur for downward sloping labor demand curve. Our second result is numerical: in a standard calibration of the model, equilibrium is determinate for empirically plausible values of the externalities and the curvature of the production possibility frontier. This result is robust to reasonable changes in the parameter values used.

The economic relevance of our findings lies in the fact that the strict concavity of the production possibility frontiers arises naturally when capital adjustment costs at the sector level are considered. In this paper, we consider a generalized form of the intratemporal capital adjustment costs suggested by Huffman and Wynne (1999).[2] One way to interpret our result therefore is that capital adjustment costs at the sector level of *any* size preclude local indeterminacy for empirically plausible parameter choices. The presence of capital adjustment costs at the sector level can be justified in three ways. First, there is substantial empirical evidence in favor of the existence of capital adjustment costs at the firm level; see Hammermesh and Pfann (1996) for a review of the evidence. Second, without capital adjustment costs at the sector level the ratio of the price of installed capital to the price of new capital ("Tobin's q") is constant over the

[1]For other versions of the neoclassical growth model, Boldrin and Rustichini (1994) and Benhabib et al. (2000) find the same difference: local indeterminacy is easier to obtain in two- than in one-sector versions. For a detailed review of this literature, see Benhabib and Farmer (1999).

[2]In a companion paper, Herrendorf and Valentinyi (2002), we consider standard intertemporal capital adjustment costs of the form suggested by Lucas and Prescott (1971). We find that they make the results obtained in this paper even stronger in that it becomes even harder to get local indeterminacy.

business cycle, which is counterfactual. Third, without capital adjustment costs at the sector level the class of two-sector models considered here has several counterfactual properties that disappear when capital adjustment costs are modeled [Huffman and Wynne (1999) and Boldrin et al. (2001)].

The intuition for our main result is closely linked to the relationship between the the composition of the capital–producing sector's output and the ratio of the relative prices of the two new capital goods. Specifically, if the production possibility frontier between the two capital goods is linear, then the relative price ratio is constant and the ratio of the new capital goods can be chosen independently of the realization of the contemporaneous relative price ratio. In contrast, if the production possibility frontier is strictly concave, then the contemporaneous relative price ratio is a function of the ratio of the new capital goods. In other words, replacing a linear production possibility frontier by a strictly concave one eliminates one degree of freedom from the model economy. Our results show that this implies that local indeterminacy becomes impossible for plausible parameter choices.

The articles most closely related to our study are Kim (1998), Wen (1998), and Guo and Lansing (2001), who study the implications of convex capital adjustment costs for the local stability properties of the one-sector neoclassical growth model with an externality. These papers have one key result in common: given a strength of increasing returns that implies local indeterminacy, there is a strictly positive, minimum size of the capital adjustment costs that makes local indeterminacy impossible. We find that costs of adjusting the sectors' capital stocks have a very different effect in the two–sector neoclassical growth model: given a strength of increasing returns that implies local indeterminacy without capital adjustment costs, introducing arbitrarily small capital adjustment costs makes local indeterminacy impossible.

The rest of the paper is organized as follows. Section 2 lays out the economic environment. Section 3 reports our analytical results. Section 4 reports our numerical results. Section 5 concludes the paper.

2 Environment

Time is continuous and runs forever. There are continua of measure one of identical, infinitely-lived households and of two types of firms. Firms of the first type produce a perishable consumption good and firms of the second type produce new capital goods. The representative household is endowed with the initial capital stocks, with the property rights for the representative firms, and with one unit of time at each instant. We assume that installed capital is sector specific, which is consistent with the evidence collected by Ramey and Shapiro (2001) that it is very costly to reallocate installed capital to other sectors. At each point in time five commodities are traded in sequential markets: the consumption good, the new capital good suitable for the production of consumption goods, the new capital good suitable for the production of new capital goods, working time in the consumption-producing sector, and working time in the capital–producing sector.

The representative household solves:

$$\max_{\{c_t,l_{ct},l_{xt},x_{ct},x_{xt},k_{ct},k_{xt}\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{\left[c_t \exp(1-l_{ct}-l_{xt})\right]^{1-\sigma}-1}{1-\sigma} dt \tag{1a}$$

$$\text{s.t.} \quad c_t + p_{ct}x_{ct} + p_{xt}x_{xt} = \pi_{ct} + \pi_{xt} + w_{ct}l_{ct} + w_{xt}l_{xt} + r_{ct}k_{ct} + r_{xt}k_{xt}, \tag{1b}$$

$$\dot{k}_{ct} = x_{ct} - \delta_c k_{ct}, \qquad \dot{k}_{xt} = x_{xt} - \delta_x k_{xt}, \tag{1c}$$

$$k_{c0} = \bar{k}_{c0} \text{ given}, \qquad k_{x0} = \bar{k}_{x0} \text{ given}, \tag{1d}$$

$$0 \le c_t, l_{ct}, l_{xt}, x_{ct}, x_{xt}, k_{ct}, k_{xt}, \qquad l_{ct} + l_{xt} \le 1. \tag{1e}$$

The notation is as follows: $\rho > 0$ is the discount rate and $\sigma \geq 0$ is the elasticity of intertemporal substitution; c_t denotes the consumption good at time t (which is the numeraire); the subscripts c and x indicate variables from the consumption-producing and the capital–producing sector; l_{ct} and l_{xt} are the working times, w_{ct} and w_{xt} are the wages, x_{ct} and x_{xt} are the new capital goods, p_{ct} and p_{xt} are the relative prices of the new capital goods, k_{ct} and k_{xt} are the capital stocks, r_{ct}

and r_{xt} are the real interest rates, δ_c and δ_x are the depreciation rates, and π_{ct} and π_{xt} are the profits (which will be zero in equilibrium).

Two features of the representative household's problem deserve further comment. First, we restrict x_{ct} and x_{xt} to be non-negative, meaning that installed capital is sector specific. Nevertheless the capital stock of a sector can be reduced by not replacing depreciated capital, so close to the steady state (the existence of which we will prove below) the non–negativity constraints will not be binding. Second, we choose the functional form for utility that is consistent with the existence of a balanced growth path and implies an infinite elasticity of labor supply.[3] The reason for focusing on infinite labor supply elasticity is that the existing studies identify this to be the best case for local indeterminacy. An economic justification for infinite labor supply elasticity is the lottery argument of Hansen (1985). As σ converges to 1, our functional form converges to $\log(c_t) + 1 - l_{ct} - l_{xt}$, which is the specifications most commonly used in the literature.[4]

Denoting by μ_{ct} and μ_{xt} the current value multipliers attached to the accumulation equations (1c), the necessary and sufficient conditions for the solution to the household's problem are (1b)–(1e) and

$$\frac{p_{ct}}{c_t} = \mu_{ct}, \qquad \frac{p_{xt}}{c_t} = \mu_{xt}, \tag{2a}$$

$$c_t = w_{ct} = w_{xt}, \tag{2b}$$

$$\dot{\mu}_{ct} \leq \mu_{ct}(\delta_c + \rho) - \frac{r_{ct}}{c_t} \qquad \text{(with equality if } x_{ct} > 0\text{)}, \tag{2c}$$

$$\dot{\mu}_{xt} \leq \mu_{xt}(\delta_x + \rho) - \frac{r_{xt}}{c_t} \qquad \text{(with equality if } x_{xt} > 0\text{)}, \tag{2d}$$

$$\lim_{t\to\infty} \frac{p_{ct}k_{ct}}{c_t} = \lim_{t\to\infty} \frac{p_{xt}k_{xt}}{c_t} = 0. \tag{2e}$$

[3] King et al. (1988) show that for a balanced growth path to exist the instantaneous utility must take the form $\frac{1}{1-\sigma}[c_t \exp(\varphi(1 - l_{ct} - l_{xt}))]^{1-\sigma}$ where φ is an increasing function.

[4] We have also experimented with a specification of the instantaneous utility that is not consistent with balanced growth, notably $\frac{c_t^{1-\sigma}-1}{1-\sigma} + 1 - l_{ct} - l_{xt}$. Our results turn out to be robust as long as σ is not chosen to be unreasonably small.

Note that, as usual, the dynamic first–order conditions (2c) and (2d) hold only for $t > 0$.

We now turn to the production side of the model economy. The problem of the representative firm of the consumption-producing sector is:

$$\max_{c_t, k_{ct}, l_{ct}} \quad \pi_{ct} \equiv c_t - r_{ct}k_{ct} - w_{ct}l_{ct} \tag{3a}$$

$$\text{s.t.} \quad c_t = A_t k_{ct}^{a} l_{ct}^{1-a}, \tag{3b}$$

$$c_t, l_{ct}, k_{ct} \geq 0, \tag{3c}$$

where $A_t \geq 0$ denotes total factor productivity in the sector and $a \in (0, 1)$. The necessary and sufficient conditions for a solution are (3b), (3c), and

$$r_{ct} = aA_t k_{ct}^{a-1} l_{ct}^{1-a}, \tag{4a}$$

$$w_{ct} = (1 - a)A_t k_{ct}^{a} l_{ct}^{-a}. \tag{4b}$$

The problem of the representative firm of the capital–producing sector is:

$$\max_{x_{xt}, x_{ct}, l_{xt}, k_{xt}} \quad \pi_{xt} \equiv p_{xt}x_{xt} + p_{ct}x_{ct} - r_{xt}k_{xt} - w_{xt}l_{xt} \tag{5a}$$

$$\text{s.t.} \quad f(x_{ct}, x_{xt}) = B_t k_{xt}^{b} l_{xt}^{1-b}, \tag{5b}$$

$$x_{xt}, x_{ct}, k_{xt}, l_{xt} \geq 0, \tag{5c}$$

where $B_t \geq 0$ denotes total factor productivity in the sector, $b \in (0, 1)$, and f is a twice continuously differentiable function that is non-negative, increasing in both arguments, linear homogeneous, and quasi-convex. Denoting the multiplier attached to (5b) by λ_t, the necessary and sufficient conditions for the solution to problem (5) are (5b), (5c), and

$$r_{xt} = \lambda_t b B_t k_{xt}^{b-1} l_{xt}^{1-b}, \tag{6a}$$

$$w_{xt} = \lambda_t (1-b) B_t k_{xt}^{b} l_{xt}^{-b}, \tag{6b}$$

$$p_{ct} \leq \lambda_t f_c(x_{ct}, x_{xt}) \qquad \text{(with equality if } x_{ct} > 0\text{)}, \tag{6c}$$

$$p_{xt} \leq \lambda_t f_x(x_{ct}, x_{xt}) \qquad \text{(with equality if } x_{xt} > 0\text{)}, \tag{6d}$$

where f_c and f_x denote the partial derivatives of f with respect to x_{ct} and x_{xt}.

The assumption of quasi-convexity implies that for given $\bar{f} \in \mathbb{R}_+$ the lower sets $\{(x_{xt}, x_{ct}) \in \mathbb{R}_+^2 | f(x_{xt}, x_{ct}) \leq \bar{f}\}$ are convex, so the production possibility frontier between the two new capital goods, x_{ct} and x_{xt}, is concave. The standard assumption in the literature is that f is linear:

$$f(x_{ct}, x_{xt}) = f_c x_{ct} + f_x x_{xt}, \tag{7}$$

where f_c and f_x are positive constants, which are often set to one.[5] If f is linear, then the production possibility frontier between the two new capital goods is linear too. Our innovation in this paper is to consider the case of non-linear, strictly quasi-convex functions f. An example is

$$f(x_{ct}, x_{xt}) = \left(f_c x_{ct}^{1+\varepsilon} + f_x x_{xt}^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}}, \tag{8}$$

where ε is a positive constant. If f is strictly quasi–convex, then the production possibility frontier between the two new capital goods becomes strictly concave. Given that capital is assumed to be sector specific, a strictly concave production possibility frontier generates capital adjustment costs. To see this, suppose that the capital stocks of both sectors are constant, so that the production of new capital goods just makes up for the depreciation of capital. Suppose then that today the capital stock in the consumption-producing sector is reduced by 1 unit.

[5] The choice of f_c and f_x amounts to a choice of the units in which x_{ct} and x_{xt} are denominated. This choice does not matter for the local stability properties of the steady state.

Since capital is sector–specific, installed capital cannot be moved across sectors. Thus, the past production of new capital goods for the consumption-producing sector is to be reduced by 1 unit. With a linear production possibility frontier, this implies that the production of capital goods for the capital–producing sector increases by $\frac{f_c}{f_x}$ units; the change in the two capital stocks is then costless. With a strictly convex production possibility frontier, this implies that the production of capital goods for the capital–producing sector increases by less than $\frac{f_c}{f_x}$ units; the change in the two capital stocks is therefore costly. It is important to realize that it remains costless to change the total output of the capital–producing sector as long as its composition is not changed. This is due to the linear homogeneity of f. Therefore, the capital adjustment costs implied by a strictly quasi–convex f are *intratemporal*, not *intertemporal*.[6]

There are several justifications for modeling capital adjustment costs at the sector level. First, there is substantial microevidence that firms' adjustment time to stochastic disturbances exceeds by far the length of one year, and hence the maximal length of a period in real business cycle models [Hammermesh and Pfann (1996)]. Second, without capital adjustment costs the ratio of the price of installed capital to the price of new capital ("Tobin's q") is constant, which is counterfactual. Third, two-sector neoclassical growth models without capital adjustment costs allow for the costless reallocation of capital across sectors. This leads to countercyclical consumption and excessive investment volatility, which are counterfactual. Huffman and Wynne (1999) show that these problems are resolved when one introduces intratemporal capital adjustment costs of the form (8).[7]

The total factor productivities are specified so that there can be positive externalities at the level of each sector:

$$A_t = k_{ct}^{\theta_c a} l_{ct}^{\theta_c(1-a)}, \qquad B_t = k_{xt}^{\theta_x b} l_{xt}^{\theta_x(1-b)}, \tag{9}$$

[6] In Herrendorf and Valentinyi (2002), we consider standard intertemporal capital adjustment costs of the form suggested by Lucas and Prescott (1971) and find that the results obtained in this paper are robust with respect to this modification.

[7] Fisher (1997) makes a related point for a two–sector neoclassical growth model with home production and market production.

where $\theta_c, \theta_x \geq 0$. Substituting (9) back into the production functions, the sectors' aggregate outputs become:

$$c_t = k_{ct}^{\alpha_1} l_{xt}^{\alpha_2}, \qquad \alpha_1 \equiv (1+\theta_c)a, \qquad \alpha_2 \equiv (1+\theta_c)(1-a), \tag{10a}$$

$$x_t = k_{xt}^{\beta_1} l_{xt}^{\beta_2}, \qquad \beta_1 \equiv (1+\theta_x)b, \qquad \beta_2 \equiv (1+\theta_x)(1-b). \tag{10b}$$

Several clarifying remarks are at order. First, (9) implies that the externalities on capital and labor are the same. The reason for this assumption is that separate estimates for the strength of the resulting increasing returns do not exist.[8] Second, conditions (6b) and (9) imply that the labor demand curve in the capital–producing sector is downward sloping for $\theta_x < \frac{b}{1-b}$ and upward sloping for $\theta_x > \frac{b}{1-b}$. Using (10b), these two conditions equivalently can be written as $\beta_2 < 1$ and $\beta_2 > 1$. Our main result will show that if the labor demand curve in the capital–producing sector slopes downward the local stability properties of the steady state are strikingly different depending on whether f is linear or strictly quasi-concave. This result is important in applications because $\theta_x > \frac{b}{1-b}$ is not plausible empirically and its implications contradict the business cycle facts [see our discussion of empirical plausible increasing returns in Section 4 and the discussion in Aiyagari (1995), respectively]. Third, the externalities are not taken into account by the firms, so a competitive equilibrium exists and in equilibrium profits are zero and the capital and labor shares are the usual ones: $\frac{r_{ct}k_{xt}}{c_t} = a$, $\frac{w_{ct}l_{xc}}{c_t} = 1-a$, $\frac{r_{xt}k_{xt}}{k_t} = b$, $\frac{w_{xt}l_{xt}}{k_t} = 1-b$. In a competitive equilibrium the total factor productivities on which the firms base their decisions must be equal to those that results from these decisions:

Definition 1 (Competitive equilibrium) *A competitive equilibrium is a collection of prices* $\{w_{ct}, w_{xt}, r_{ct}, r_{xt}, p_{ct}, p_{xt}\}_{t=0}^{\infty}$, *allocations* $\{c_t, l_{ct}, l_{xt}, x_{ct}, x_{xt}, k_{ct}, k_{xt}\}_{t=0}^{\infty}$, *and total factor productivities* $\{A_t, B_t\}_{t=0}^{\infty}$ *such that: (i)* $\{c_t, l_{ct}, l_{xt}, x_{ct}, x_{xt}, k_{ct}, k_{xt}\}_{t=0}^{\infty}$ *solve the problem of the representative*

[8] The results of Harrison and Weder (2001) suggest that imposing this constraint does not affect the stability properties of the steady state of the two-sector neoclassical growth model without capital adjustment costs in an important way.

household, (1)*; (ii)* $\{c_t, l_{ct}, k_{ct}\}_{t=0}^{\infty}$ *solve the problem of the representative firm of the consumption-producing sector,* (3)*; (iii)* $\{x_{xt}, x_{ct}, l_{xt}, k_{xt}\}_{t=0}^{\infty}$ *solve the problem of the representative firm of the capital–producing sector,* (5)*; (iv)* A_t *and* B_t *are determined consistently, that is, the two equations in* (9) *hold.*[9]

3 Analytical Results

3.1 Local stability properties

We start by establishing that there is a unique steady state and by deriving the reduced-form equilibrium dynamics nearby.

Proposition 1 (Reduced–form dynamics)

(i) There is a unique steady state.

(ii) If f *is linear, then there is a neighborhood of the steady state such that the equilibrium reduced–form dynamics can be described by the dynamics of the state variable* $k_t \equiv f_c k_{ct} + f_x k_{xt}$ *and the dynamics of the control variable* μ_{ct}.

(iii) If f *is strictly quasi convex, then there is a neighborhood of the steady state such that the equilibrium reduced–form dynamics can be described by the dynamics of the two state variables* k_{ct} *and* k_{xt} *and the two control variables* μ_{ct} *and* μ_{xt}.

Proof. See the Appendix A.

The proposition shows that the equilibrium reduced–form dynamics close to the steady state are two dimensional when f is linear and four dimensional when f is strictly quasi convex. The reason for this difference is as follows. With a linear f the ratio between the shadow prices of the two new capital stocks is constant, implying that only one of them is needed. Moreover,

[9] Note that since we have two sectors here, market clearing is automatically satisfied when the firms' production constraints are satisfied. Thus, we do not need to specify an economy-wide resource constraint.

the ratio of the two shadow prices at any point in time does not determine the composition of the capital–producing sector's output of new capital goods. Consequently, if $t > 0$ any allocation of the aggregate capital stock across the two sectors can be achieved by choosing the right composition of the past outputs of the capital producing–sector, implying that only the aggregate capital stock is needed as a state. Note that for this to be the case, the model economy needs to be close to the steady state where the non-negativity constraints on the two new capital goods do not bind because the two existing capital stocks can be reduced by as much as desired by not replacing depreciated capital.[10]

With a strictly quasi–convex f, the equilibrium reduced–form dynamics are four dimensional for the following reasons. First, the ratio of the shadow prices of the two new capital goods is not constant, implying that both shadow prices are needed to describe the dynamics. Second, the ratio of the two shadow prices at any point in time uniquely determines the composition of the capital–producing sector's output of new capital goods at that point in time. This follows from the fact that combining (2a), (6c), and (6d) (the last two with equality) results in

$$\frac{\mu_{ct}}{\mu_{xt}} = \frac{f_c\left(\frac{x_{ct}}{x_{xt}},1\right)}{f_x\left(\frac{x_{ct}}{x_{xt}},1\right)}. \tag{11}$$

Consequently, the capital stocks of both sectors become state variables.

We now explore analytically the stability properties of the steady state. The steady state is saddle–path stable if there are as many stable roots (i.e. roots with negative real part) as states and as many unstable roots (i.e. roots with positive real part) as controls. The steady state is stable if there are more stable roots than states and it is unstable if there more unstable roots than controls. If the steady state is saddle–path stable then the equilibrium is determinate, that is, given the initial capital stocks close to the steady state values there are unique initial

[10]Note also that this argument does not hold at $t = 0$ when both k_{c0} and k_{x0} are given because installed capital is assumed to be sector–specific. This is different from the version of the two–sector model in which capital is not sector–specific. In this case, only the aggregate capital stock matters at $t = 0$. Christiano (1995) shows that this difference does not matter for the stability properties of the steady state.

shadow prices such that the model economy converges to the steady state. If the steady state is stable, then the equilibrium is locally indeterminate, that is, given the initial capital stocks close to the steady state values there exists a continuum of shadow prices such that the model economy converges to the steady state. Since it is not feasible to compute analytically the four eigenvalues, we will only compute the determinant and the trace of the linearization of the reduced-form equilibrium dynamics at the steady state. Although this does not allow for a full characterization of the local stability properties, it provides important information because the determinant equals the product of the eigenvalues and the trace equals the sum of the real parts of the eigenvalues (complex eigenvalues occur in conjugates, implying that the imaginary parts cancel in the summation). This leads to the next proposition, which constitutes the main result of our paper.

Proposition 2 (Local stability properties of the steady state) *Suppose that* $b < 1-b$ *and* $\theta_x \in [0, \frac{1-b}{b})$.

(i) Suppose that f *is linear.*

There are constants $\underline{\theta}_x \in (0, \frac{b}{1-b})$ *and* $\overline{\theta}_x \in (-\frac{\rho b}{\rho b+\delta_x}, \frac{1-b}{b})$ *such that:*

(i.a) if

$$(\rho + \delta_x)[\rho + (1-b)\delta_x] > \frac{\rho b+\delta_x}{\rho} b\delta_c(\rho + \delta_c),$$

then $\underline{\theta}_x < \overline{\theta}_x$ *and the steady state is saddle–path stable for* $\theta_x \in [0, \underline{\theta}_x)$, *stable for* $\theta_x \in (\underline{\theta}_x, \overline{\theta}_x)$, *and unstable for* $\theta_x \in (\overline{\theta}_x, \frac{1-b}{b})$*;*

(i.b) if

$$(\rho + \delta_x)[\rho + (1-b)\delta_x] < \frac{\rho b+\delta_x}{\rho} b\delta_c(\rho + \delta_c),$$

then $\overline{\theta}_x < 0 < \underline{\theta}_x$ *and the steady state is saddle–path stable for* $\theta_x \in [0, \underline{\theta}_x)$ *and stable for* $\theta_x \in (\underline{\theta}_x, \frac{1-b}{b})$.

(ii) Suppose that f *is strictly quasi–convex.*

(ii.a) $\theta_x \in [0, \frac{b}{1-b})$ *is a necessary condition for the steady state to be saddle–path stable;*

(ii.b) $\theta_x \in (\frac{b}{1-b}, \frac{1-b}{b})$ *is a necessary condition for the steady state to be stable.*

Proof. See the Appendix B.

We begin the discussion of our main results by noting that calibrations of our two-sector model that are typical in the business cycle and growth literature are consistent with the assumptions $b < 1 - b$ and $\theta_x < \frac{1-b}{b}$.[11] The inequality $b < 1 - b$ ensures that the capital share in the capital–producing sector's income is smaller than one half and that $\frac{b}{1-b} < \frac{1-b}{b}$. The inequality $\theta_x < \frac{1-b}{b}$ ensures that the aggregate returns to capital are less than one so there cannot be endogenous growth in steady state. As pointed out before, there are two relevant subcases of $\theta_x < \frac{1-b}{b}$: for $\theta_x \in [0, \frac{b}{1-b})$ the labor demand curve of the capital–producing sector slopes downward and for $\theta_x \in (\frac{b}{1-b}, \frac{1-b}{b})$ it slopes upward.

We continue the discussion of this proposition with the case of a linear f (part (i) of the proposition). It says that if the labor demand curve in the capital–producing sector slopes downward, then the steady state can be saddle–path stable, stable, or unstable.[12] The key part of this statement is that a linear f allows for a stable steady state and therefore for local indeterminacy at the steady state when the labor demand curve in the capital–producing sector slopes *downward*. This replicates the result of the recent literature on self–fulfilling business cycles; see for example Benhabib and Farmer (1996) and Harrison and Weder (2001).

We conclude the discussion of our results with the case of a strictly–quasi convex f (part (ii) of the proposition). It says that if the labor demand curve in the capital–producing sector slopes *upward*, then the steady state can be stable or unstable but not saddle–path stable; if the labor demand curve in the capital–producing sector slopes *downward*, then the steady state can be saddle–path stable or unstable but not stable. Thus, a strictly–quasi convex f rules out local indeterminacy at the steady state if labor demand curve slopes downward. This is our key

[11]Below we will discuss calibration issues in more detail.

[12]It is easy verify using the results from Appendix B that if $\rho[\rho + (1-b)\delta_x] > b\delta_c(\rho + \delta_c)$, then $\overline{\theta} < \frac{b}{1-b}$ and the steady state can be unstable under downward sloping labor demand curve.

analytical result, which holds for *any* strictly quasi–convex f, and thus for intratemporal capital adjustment costs of *any* positive size. In other words, the local stability properties of the two–sector neoclassical growth model with strictly quasi–convex f differ strikingly from those with a linear f. In fact, the local stability properties of the two–sector real business cycle model with strictly quasi–convex f are much more like those of the one-sector neoclassical growth model without capital adjustment costs, in which local indeterminacy requires an upward-sloping labor demand curve [Benhabib and Farmer (1994)].

3.2 Intuition

Here we seek to understand why a strictly quasi–convex f precludes the possibility of local indeterminacy for moderate externalities that leave the labor demand of the capital–producing sector downward sloping. We start by demonstrating that as the model economies with strictly quasi–convex f converge to that with a linear f, the steady states behave continuously. So a discontinuity at the steady state cannot be the explanation of our results. In order to be able to establish this, we need to specify what we mean by convergence.

Assumption 1 (Convergence to a linear *f*) *Consider a linear function* $f : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$ *with* $f(x_{ct}, x_{xt}) = f_c x_{ct} + f_x x_{xt}$ *where* $f_c,\ f_x \geq 0$, *denote the steady state values of the new capital goods in the associated model economy by* (x_c, x_x), *and let* $U(x_c, x_x)$ *be a small open neighborhood of* (x_c, x_x). *Furthermore, consider a sequence* $\{f_i\}_{i=1}^{\infty}$ *of functions* $f_i : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$ *that are non-negative, linear homogeneous, twice continuously differentiable, and strictly quasi-convex.*

We say that $\{f_i\}_{i=1}^{\infty}$ *converges to* f *on* $U(x_c, x_x)$ *if and only if each of* $\{f_i\}_{i=1}^{\infty}$, $\{f_{c,i}\}_{i=1}^{\infty}$, $\{f_{x,i}\}_{i=1}^{\infty}$, $\{f_{cc,i}\}_{i=1}^{\infty}$, $\{f_{xx,i}\}_{i=1}^{\infty}$, *and* $\{f_{cx,i}\}_{i=1}^{\infty}$ *converge in the supremum norm defined over* $U(x_c, x_x)$ *to* f, f_c, f_x, f_{cc}, f_{xx}, *and* f_{cx}, *respectively.*

Proposition 3 (Continuity of the steady states) *Consider a sequence of functions* $\{f_i\}_{i=1}^{\infty}$ *of the form assumed in Assumption 1. Then the sequence of the steady states of the economies with* f_i *converges to the steady state of the model economy with* f.

Proof. See the Appendix C.

To find the explanation for our results, it is useful to recall how local indeterminacy can occur for mild strengths of the externality with a linear f.[13] So suppose the model economy is in steady state and ask whether there can be other equilibrium paths with the same initial capital stocks equal to their steady state values but temporarily higher capital stocks subsequently. For such paths to exist, the compositions of the capital outputs need to be changed. In particular, more capital goods for the capital–producing sector need to be produced initially and fewer capital goods for this sector need to be produced subsequently. The resulting higher and then a lower capital stocks in the capital producing-sector imply that consumption growth is first lower and then higher. In order to make such paths optimal for the representative household there must first be higher and then lower returns on capital. They can come about because the aggregate production possibility frontier (henceforth PPF) between consumption and composite capital goods is strictly convex and the relative price of the two new capital goods is constant irrespective of the composition of new capital output. Specifically, given these features, lower (higher) consumption–to–capital–goods ratios are associated with lower (higher) relative prices of capital goods in terms of consumption, so capital gains result that generate the required movements of the returns to capital.

Two crucial ingredients bring about the capital gains when f is linear. First, the production possibility frontier between new capital goods and consumption is strictly convex at the steady state. This ingredient is also present when f is strictly quasi–convex but almost linear.

Proposition 4 (Continuity of the PPF) *Consider a sequence of functions* $\{f_i\}_{i=1}^{\infty}$ *of the form assumed in Assumption 1. Providing* $x_{ct}, x_{xt} > 0$*, the sequence of the production possibility frontiers of the economies with* f_i *converges on* $U(x_c, x_x)$ *to the production possibility frontier of the model economy with* f*.*

Proof. See the Appendix D.

[13] This follows Christiano (1995).

The second crucial ingredient that brings about the capital gains when f is linear is that the composition of new capital goods can be changed independently of the relative price of the two capital goods, $\frac{p_{xt}}{p_{ct}}$. Formally this follows from the fact that with a linear f the ratio between the two capital goods, $\frac{x_{xt}}{x_{ct}}$, is not determined by the realizations of the other variables up to date t. This second ingredient is not present when f is strictly quasi convex — instead, the relative price ratio at date t uniquely determines the investment ratio:

$$\frac{x_{xt}}{x_{ct}} = g\left(\frac{\mu_{xt}}{\mu_{ct}}\right) = g\left(\frac{p_{xt}}{p_{ct}}\right). \tag{12}$$

To see how this rules out alternative paths, note that (12) implies that the initial increase in the production of capital goods for the capital–producing sectors is now associated with an *increase* in $\frac{p_{xt}}{p_{ct}}$. Therefore, to make the representative household indifferent between holding the two capital goods, there now needs to be a stronger capital gain on capital goods for the capital–producing sector than on those for the consumption–producing sector. From (12) this must be associated with a further increase in the production of capital goods for the capital–producing sector relative to the other ones. As a result, such alternative paths can never return to the steady state and so violate the transversality condition.

It should be pointed out that this intuition works for arbitrarily small effects of changes in $\frac{x_{xt}}{x_{ct}}$ on $\frac{p_{xt}}{p_{ct}}$, and thus for arbitrarily small intratemporal capital adjustment costs. It should also be pointed out that this intuition carries over to models with more than two sectors and more than two capital goods. Small intratemporal capital adjustment costs between any pair of capital goods would then also determine uniquely the ratio of any two different capital goods. Since this is the key mechanism behind our result, we conjecture that arbitrarily small capital adjustment costs could also be used to obtain determinacy when there are more than two sectors.

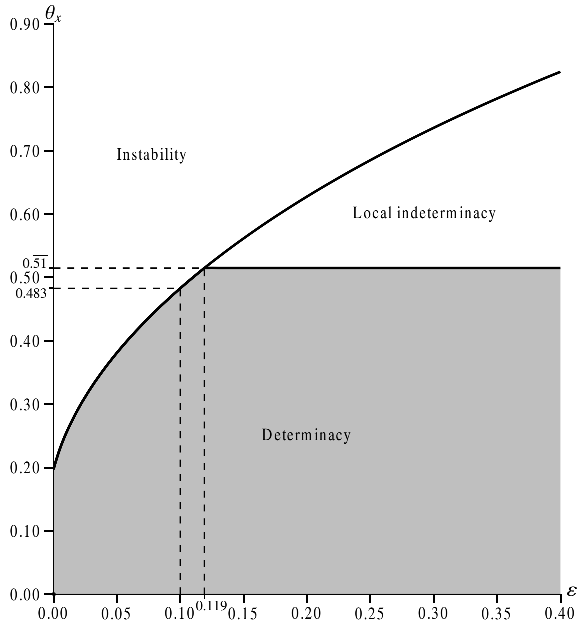
4 Numerical Results

The analytical results derived so far for strictly quasi–convex functions f do not settle the question whether the steady state of the class of two-sector neoclassical growth models with sector–specific externalities is saddle–path stable or unstable for reasonable parameter choices. To answer this question we now calibrate the model and then compute numerically the four eigenvalues that determine the stability properties of the steady state. We use the functional forms and the parameter values of Huffman and Wynne (1999), who calibrate a two-sector model similar to our's but with constant returns in both sectors, so $\theta_c = \theta_x = 0$ in their model. This difference does not affect the usefulness of their calibration for our purposes because the degrees of increasing returns do not affect the calibration of the other parameters. The specific assumptions of Huffman and Wynne are that $\sigma = 1$ (so the period–utility in consumption is logarithmic) and that f is of the form (8). Using quarterly, postwar, one-digit US data, Huffman and Wynne calibrate $\delta_c = 0.018$, $\delta_x = 0.020$, $a = 0.41$, $b = 0.34$, and $\rho = 0.01$. Moreover, they calibrate $\varepsilon = 0.1$ or $\varepsilon = 0.3$, depending on the procedure.[14]

The equations for the linearized reduced–form equilibrium dynamics with strictly quasi–convex f, (B.7)–(B.9), show that the dynamics are independent of θ_c, so we need to choose a value only for θ_x. The available evidence on increasing returns is rather mixed. However, it is non-controversial that Hall's (1988) initial estimates of aggregate increasing returns of about 0.5 were upward biased. More recent empirical studies instead find estimates between constant returns and milder increasing returns up to 0.3; see e.g. Bartelsman et al. (1994), Burnside et al. (1995), or Basu and Fernald (1997). According to Basu and Fernald (1997) these aggregate increasing returns are mainly due to increasing returns in the capital–producing sector; specifically they estimate non-durable manufacturing to have constant returns and durable manufacturing to have increasing returns up to 0.36. Since θ_x is a key parameter determining the local stability properties of the steady state and since it is hard to draw a sharp line between

[14]See their paper for the details.

Figure 1: Local stability for *intratemporal* adjustment costs and $\sigma = 1$, $\rho = 0.01$, $\delta_c = 0.018$, $\delta_x = 0.020$, $a = 0.41$, $b = 0.34$.



empirically plausible and implausible values for it, we will vary it extensively together with the other key parameter ε. Specifically, we will explore the local stability properties of the steady state for all $\theta_x \in (0.000, 0.900)$ and $\varepsilon \in (0.000, 0.400)$.[15]

Our numerical results are reported in Figure 1. They confirm the analytical result of Proposition 2 that an upward-sloping (downward-sloping) labor demand curve in the capital–producing is a necessary condition for local indeterminacy (determinacy).[16] Our numerical results go beyond the analytical ones in three respects. First, they show that, given the calibration used here, an upward-sloping (downward-sloping) labor demand curve in the capital–producing becomes a sufficient condition for local indeterminacy (determinacy) when the intratemporal capital adjustment costs are sufficiently large ($\varepsilon \geq 0.119$). Second, they show that, given the calibration used here and capital adjustment costs within the range calibrated by Huffman and Wynne,

[15]$\varepsilon = 0.000000001$ is the value closest to zero that we try in these computations.

[16]Given the calibration used here, the labor demand curve slopes upward if and only if $\theta_x > 0.\overline{51}$.

$\varepsilon \in [0.1, 0.3]$, the steady state is determinate if the increasing returns do not exceed 0.483. The range $\theta_x \in [0, 0.483]$ includes all values of increasing returns that are usually considered reasonable. So, given $\varepsilon \in [0.1, 0.3]$, the local stability properties with a strictly quasi–convex f are summarized by determinacy for every empirically plausible specification of θ_x. Third, our numerical results show that, given the calibration used here, arbitrarily small capital adjustment costs make the equilibrium determinate for $\theta_x \in (0, 0.197)$, whereas Proposition 2 shows that without capital adjustment costs the equilibrium is locally indeterminate for $\theta_x \in (0.072, 0.197)$. Thus, the steady state with small capital adjustment cost is saddle–path stable in the region of increasing returns in which the steady state without them is stable.[17]

It should be pointed out that there is a possibility for global indeterminacy. This follows from the additional piece of information that at the bifurcation to "instability" two of the eigenvalues are complex and their real parts change sign, that is, a Hopf bifurcation occurs. The Hopf bifurcation theorem implies the existence of limit cycles, which may or may not be stable. If they are stable, then a form of global indeterminacy occurs. Since the Hopf bifurcation does not occur for plausible parameter values, we do not study this issue.

We complete this section with a brief discussion of the robustness of our numerical findings, which we have explored in two directions. First, we have shown that our numerical determinacy result survives for reasonable variations of the parameter values used above. The details of this sensitivity analysis are reported in a technical appendix that is available upon request. Second, we have shown that our numerical determinacy result survives for the intertemporal capital adjustment costs of the form suggested by Lucas and Prescott (1971). In fact, it turns out that these intertemporal capital adjustment costs make it even harder to get local indeterminacy than the intratemporal ones considered in this paper. The details can be found in Herrendorf and Valentinyi (2002).

[17] This third result has a similar flavor as the recent result of Shannon and Zame (2002), who show that ruling out preferences with perfect substitutability between different consumption goods can bring about determinacy in an exchange economy.

5 Conclusion

We have explored the conditions under which indeterminacy of equilibrium occurs nearby the steady state in a class of two-sector neoclassical growth models with sector–specific externalities. Our main finding has been that a strictly concave production possibility frontier between the two new capital goods, which captures intratemporal capital adjustment costs, precludes local indeterminacy for every empirically plausible specification of the model parameters. This analytical result contrasts sharply with the standard result that with a linear production possibility frontier, local indeterminacy can occur in the two-sector model for a wide range of plausible parameter values. It can be interpreted to mean that local indeterminacy is not a robust property of the class of two-sector neoclassical growth models with sector–specific externalities. We conjecture that this result is likely to carry over to models with more than two sectors and more than two capital goods.

Our findings are relevant for several reasons. To begin with, if local indeterminacy is impossible for plausible specifications of the parameter values, then self-fulfilling business cycles are impossible for plausible specifications of the parameter values. Since self-fulfilling business cycles are often inefficient whereas business cycles driven by fundamental shocks are often efficient, this has important implications for the debate about whether or not government policy should aim to stabilize business cycles. Second, models from the class of two-sector neoclassical growth models that we have studied here are widely used; see for example Fisher (1997), Huffman and Wynne (1999), and Boldrin et al. (2001). Our results provide a better understanding of the local stability properties of this important class of models. Finally, our study contributes to a recent debate about the robustness of multiple and indeterminate equilibria. Even though Morris and Shin (1998) and Herrendorf et al. (2000) studied rather different environments with externalities, they share a common theme with the present paper: the introduction of frictions can substantially reduce the scope for the multiplicity or local indeterminacy of equilibrium.

Appendix

A Proof of Proposition 1

A.1 Strictly quasi–convex f

A.1.1 Reduced-form dynamics

Suppose that all first-order conditions hold with equality. (1c) and (2b)-(2d) then imply

$$\dot{k}_{ct} = x_{ct} - \delta_c k_{ct}, \qquad \dot{k}_{xt} = x_{xt} - \delta_x k_{xt}, \tag{A.1a}$$

$$\dot{\mu}_{ct} = \mu_{ct}(\delta_c + \rho) - \frac{r_{ct}}{w_{ct}}, \qquad \dot{\mu}_{xt} = \mu_{xt}(\delta_x + \rho) - \frac{r_{xt}}{w_{xt}}. \tag{A.1b}$$

To represent the model economy as a dynamical system in k_{ct}, k_{xt}, μ_{ct}, and μ_{xt}, we need to express all endogenous variables, i.e. x_{ct}, x_{xt}, l_{ct}, l_{xt}, r_{ct}, r_{xt}, p_{ct}, p_{xt}, w_{ct}, and w_{xt}, as functions of these four variables. Establishing this is the first step of the proof.

To begin with, note that (2a) implies that $\frac{p_{ct}}{p_{xt}} = \frac{\mu_{ct}}{\mu_{xt}}$, so (6c) and (6d) (with equality) together with the strict quasi–convexity of t imply that there is a function g such that:

$$g\left(\frac{\mu_{ct}}{\mu_{xt}}\right) \equiv \left(\frac{f_c}{f_x}\right)^{-1}\left(\frac{\mu_{ct}}{\mu_{xt}}\right) = \frac{x_{ct}}{x_{xt}}. \tag{A.2a}$$

Next, observe that dividing (4a) by (4b) and (6a) by (6b) and using (A.3a), we can express the factor price ratios as functions of the corresponding factors:

$$\frac{r_{ct}}{w_{ct}} = \frac{a}{1-a}\frac{l_{ct}}{k_{ct}} \qquad \frac{r_{xt}}{w_{xt}} = \frac{b}{1-b}\frac{l_{xt}}{k_{xt}}. \tag{A.2b}$$

Now, we derive labor in the consumption-producing sector. Combining (2b), (3b) and (4b) gives:

$$l_{ct} = 1 - a. \tag{A.3a}$$

Turning to labor in the capital–producing sector, observe that (2b) implies $1 = \mu_{xt}\frac{w_{xt}}{p_{xt}}$. Substituting (6b) and (6b) into this leads to

$$1 = (1-b)\mu_{xt}k_{xt}^{\beta_1}l_{xt}^{\beta_2-1}\left[f_x\left(g\left(\frac{\mu_{ct}}{\mu_{xt}}\right),1\right)\right]^{-1},$$

where we used the fact that $f(\cdot,\cdot)$ is homogeneous of degree one, and (A.2a). Rearranging leads to the reduced form for labor in the capital–producing sector:

$$l_{xt} = l_x(k_{xt},\mu_{ct},\mu_{xt}) \equiv [(1-b)\mu_{xt}]^{\frac{1}{1-\beta_2}} f_x\left(g\left(\frac{\mu_{ct}}{\mu_{xt}}\right),1\right)^{\frac{1}{\beta_2-1}} k_{xt}^{\frac{\beta_1}{1-\beta_2}}. \tag{A.3b}$$

Substituting (A.3a) and (A.3b) into (A.2b) for l_c and l_x, rearranging and plugging the result into (A.1b) gives:

$$\dot{\mu}_{ct} = F_{\mu c}(k_{ct},k_{xt},\mu_{ct},\mu_{xt}) \equiv (\rho+\delta_c)\mu_{ct} - \frac{a}{k_{ct}}, \tag{A.4a}$$

$$\dot{\mu}_{xt} = F_{\mu x}(k_{ct},k_{xt},\mu_{ct},\mu_{xt}) \equiv (\rho+\delta_x)\mu_{xt} - \frac{b}{1-b}[(1-b)\mu_{xt}]^{\frac{1}{1-\beta_2}} f_x\left(g\left(\frac{\mu_{ct}}{\mu_{xt}}\right),1\right)^{\frac{1}{\beta_2-1}} k_{xt}^{\frac{\beta_1+\beta_2-1}{1-\beta_2}}. \tag{A.4b}$$

Next, we derive the expressions for each type of investment. Substituting (9) and (A.2a) into (5b) gives

$$k_{xt}^{\beta_1}l_{xt}^{\beta_2} = x_{ct}\frac{f\left(g\left(\frac{\mu_{ct}}{\mu_{xt}}\right),1\right)}{g\left(\frac{\mu_{ct}}{\mu_{xt}}\right)} = x_{xt}f\left(g\left(\frac{\mu_{ct}}{\mu_{xt}}\right),1\right).$$

To eliminate l_{xt} from these expressions, we use (A.3b). Solving afterwards for x_{ct} and x_{xt} gives:

$$x_{ct} = x_c(k_{xt},\mu_{ct},\mu_{xt}) \equiv [(1-b)\mu_{xt}]^{\frac{\beta_2}{1-\beta_2}} \frac{g\left(\frac{\mu_{ct}}{\mu_{xt}}\right) f_x\left(g\left(\frac{\mu_{ct}}{\mu_{xt}}\right),1\right)^{\frac{\beta_2}{\beta_2-1}}}{f\left(g\left(\frac{\mu_{ct}}{\mu_{xt}}\right),1\right)} k_{xt}^{\frac{\beta_1}{1-\beta_2}},$$

$$x_{xt} = x_x(k_{xt},\mu_{ct},\mu_{xt}) \equiv [(1-b)\mu_{xt}]^{\frac{\beta_2}{1-\beta_2}} \frac{f_x\left(g\left(\frac{\mu_{ct}}{\mu_{xt}}\right),1\right)^{\frac{\beta_2}{\beta_2-1}}}{f\left(g\left(\frac{\mu_{ct}}{\mu_{xt}}\right),1\right)} k_{xt}^{\frac{\beta_1}{1-\beta_2}}.$$

Substituting the above reduced forms for x_{ct}, x_{xt}, into (A.1a) and rearranging, we find the reduced–form equilibrium dynamics:

$$\dot{k}_{ct} = F_{kc}(k_{ct}, k_{xt}, \mu_{ct}, \mu_{xt}) \equiv [(1-b)\mu_{xt}]^{\frac{\beta_2}{1-\beta_2}} \frac{g\left(\frac{\mu_{ct}}{\mu_{xt}}\right) f_x\left(g\left(\frac{\mu_{ct}}{\mu_{xt}}\right),1\right)^{\frac{\beta_2}{\beta_2-1}}}{f\left(g\left(\frac{\mu_{ct}}{\mu_{xt}}\right),1\right)} k_{xt}^{\frac{\beta_1}{1-\beta_2}} - \delta_c k_{ct}, \tag{A.4c}$$

$$\dot{k}_{xt} = F_{kx}(k_{ct}, k_{xt}, \mu_{ct}, \mu_{xt}) \equiv [(1-b)\mu_{xt}]^{\frac{\beta_2}{1-\beta_2}} \frac{f_x\left(g\left(\frac{\mu_{ct}}{\mu_{xt}}\right),1\right)^{\frac{\beta_2}{\beta_2-1}}}{f\left(g\left(\frac{\mu_{ct}}{\mu_{xt}}\right),1\right)} k_{xt}^{\frac{\beta_1}{1-\beta_2}} - \delta_x k_{xt}. \tag{A.4d}$$

A.1.2 Existence and uniqueness of steady state

Representing variables in steady state by dropping the time index t and assuming that all first-order conditions hold with equality, the steady state versions of (A.4b) and (A.4d) are found to be:

$$\delta_x k_x^{\frac{1-\beta_1-\beta_2}{1-\beta_2}} = [(1-b)\mu_x]^{\frac{\beta_2}{1-\beta_2}} \frac{f_x\left(g\left(\frac{\mu_c}{\mu_x}\right),1\right)^{\frac{\beta_2}{\beta_2-1}}}{f\left(g\left(\frac{\mu_c}{\mu_x}\right),1\right)}, \tag{A.5a}$$

$$(\rho + \delta_x) k_x^{\frac{1-\beta_1-\beta_2}{1-\beta_2}} = b[(1-b)\mu_x]^{\frac{\beta_2}{1-\beta_2}} f_x\left(g\left(\frac{\mu_c}{\mu_x}\right),1\right)^{\frac{1}{\beta_2-1}}. \tag{A.5b}$$

Dividing the second equation by the first one leads to

$$\frac{\rho+\delta_x}{b\delta_x} = \frac{f\left(g\left(\frac{\mu_c}{\mu_x}\right),1\right)}{f_x\left(g\left(\frac{\mu_c}{\mu_x}\right),1\right)}. \tag{A.6}$$

Given the assumed properties of f, this expression can be solved uniquely for $\frac{\mu_c}{\mu_x}$, so the steady state shadow price ratio is uniquely determined by the parameters of the model. From now on we will therefore write f, f_x, and g for the unique steady state values of these functions. We can then write (A.4a), (A.4c), and (A.4d) evaluated at the steady state as follows:

$$\mu_{ct} = \frac{a}{\rho+\delta_c} k_{ct}^{-1}, \tag{A.7a}$$

$$\delta_c k_c = \frac{[(1-b)\mu_x]^{\frac{\beta_2}{1-\beta_2}} g f_x^{\frac{\beta_2}{\beta_2-1}}}{f} k_x^{\frac{\beta_1}{1-\beta_2}}, \tag{A.7b}$$

$$\delta_x k_x = \frac{[(1-b)\mu_x]^{\frac{\beta_2}{1-\beta_2}} f_x^{\frac{\beta_2}{\beta_2-1}}}{f} k_x^{\frac{\beta_1}{1-\beta_2}}. \tag{A.7c}$$

To show uniqueness, we will show that k_c, μ_x, and μ_c are functions of k_x. We will then show that k_x is uniquely determined by the parameters of the model. Dividing (A.7b) by (A.7c) gives k_c as a function of k_x:

$$k_c = \frac{\delta_x}{\delta_c g} k_x. \tag{A.8}$$

Since from (A.7a) μ_c is a function of k_c, (A.8) implies that μ_c is a function of k_x. Since from (A.6) μ_x is a function of μ_c, (A.8) implies that μ_x is a function of k_x. Finally, substituting $\mu_x(k_x)$ into (A.7c), we find that k_x is uniquely determined by the parameters of the model.

We complete this part of the proof by noting that the non-negativity constraints on the investment goods are not binding in either steady state, because $x_i = \delta_i k_i$ is strictly positive for $\delta_i \in (0, 1)$. This justifies the above assumption that all first-order conditions hold with equality at the steady state. This also implies that there will be neighborhood of the steady state in which all first-order conditions hold with equality.

A.2 Linear *f*

A.2.1 Reduced-form dynamics

Assuming interior solutions and following the same steps as before, one can show that with a linear f the equilibrium dynamics are characterized by the following equations:

$$\dot{k}_{ct} = x_{ct} - \delta_c k_{ct}, \qquad \dot{k}_{xt} = x_{xt} - \delta_x k_{xt}, \qquad k_{xt}^{\beta_1} l_{xt}^{\beta_2} = f_c x_{ct} + f_x x_{xt}, \tag{A.9a}$$

$$l_{ct} = (1-a), \qquad l_{xt} = \left[\frac{(1-b)\mu_{xt}}{f_x}\right]^{\frac{1}{1-\beta_2}} k_{xt}^{\frac{\beta_1}{1-\beta_2}}, \tag{A.9b}$$

$$\frac{\mu_{ct}}{\mu_{xt}} = \frac{f_c}{f_x}, \qquad \dot{\mu}_{ct} = \mu_{ct}(\rho+\delta_c) - \frac{a}{1-a}\frac{l_{ct}}{k_{ct}}, \qquad \dot{\mu}_{xt} = \mu_{xt}(\rho+\delta_x) - \frac{b}{1-b}\frac{l_{xt}}{k_{xt}}. \tag{A.9c}$$

If none of the non-negativity constraints on x_{ct} and x_{xt} binds, then we can reduce these equations to three equations in k_{ct}, k_{xt}, and μ_{ct} that describe the reduced–form equilibrium dynamics:

$$f_c\dot{k}_{ct} + f_x\dot{k}_{xt} = \left[\frac{(1-b)\mu_{ct}}{f_c}\right]^{\frac{\beta_2}{1-\beta_2}} k_{xt}^{\frac{\beta_1}{1-\beta_2}} - f_c\delta_c k_{ct} - f_x\delta_x k_{xt}, \tag{A.10a}$$

$$\dot{\mu}_{ct} = \mu_{ct}(\rho+\delta_c) - \frac{a}{k_{ct}}, \tag{A.10b}$$

$$0 = f_x\mu_{ct}(\delta_x - \delta_c) + \frac{a f_x}{k_{ct}} - \frac{f_c b}{1-b}\left[\frac{(1-b)\mu_{ct}}{f_c}\right]^{\frac{1}{1-\beta_2}} k_{xt}^{\frac{\beta_1+\beta_2-1}{1-\beta_2}}. \tag{A.10c}$$

Note that unlike for a strictly quasi–convex f, we cannot analytically reduce these three equations to two equations that characterize fully the reduced–form equilibrium dynamics.

A.2.2 Existence and uniqueness of steady state

In steady state, the three equations in (A.10) become:

$$0 = \left[\frac{(1-b)\mu_c}{f_c}\right]^{\frac{\beta_2}{1-\beta_2}} k_x^{\frac{\beta_1}{1-\beta_2}} - f_c\delta_c k_c - f_x\delta_x k_x, \tag{A.11a}$$

$$0 = \mu_c(\rho + \delta_c) - \frac{a}{k_c}, \tag{A.11b}$$

$$0 = f_x\mu_c(\delta_x - \delta_c) + \frac{a f_x}{k_c} - \frac{f_c b}{1-b}\left[\frac{(1-b)\mu_c}{f_c}\right]^{\frac{1}{1-\beta_2}} k_x^{\frac{\beta_1+\beta_2-1}{1-\beta_2}}. \tag{A.11c}$$

The existence and uniqueness of the steady state can be shown as follows. First, (A.11b) implies that k_c is a function of μ_c. Second, substituting the result into (A.11c) implies that k_x too is a function of μ_c. Third, substituting these two expressions into (A.11a) and rearranging gives the steady state value for μ_c. Finally, (A.9) shows that all other steady state variables are functions of k_c, k_x, and μ_c.

We complete the proof by noting that the non-negativity constraints on the investment goods are not binding in either steady state, because $x_i = \delta_i k_i$ is strictly positive for $\delta_i \in (0, 1)$. This justifies the above assumption that all first-order conditions hold with equality at the steady state. This also implies that there will be neighborhood of the steady state in which all first-order conditions hold with equality.

B Proof of Proposition 2

B.1 Linear f

B.1.1 Computation of the determinant and the trace

We start with the linearization of (A.10) at the steady state:

$$\begin{aligned}
\dot{k}_t &= \tfrac{\beta_2}{1-\beta_2}\tfrac{f_c\delta_c k_c+f_x\delta_x k_x}{\mu_c}(\mu_{ct}-\mu_c)+\left[\tfrac{\beta_1}{1-\beta_2}\tfrac{f_c\delta_c k_c+f_x\delta_x k_x}{k_x}-f_x(\delta_x-\delta_c)\right](k_{xt}-k_x)-\delta_c(k_t-k),\\
\dot{\mu}_{ct} &= (\rho+\delta_c)(\mu_{ct}-\mu_c)-\tfrac{f_x(\rho+\delta_c)\mu_c}{f_c k_c}(k_{xt}-k_x)+\tfrac{(\rho+\delta_x)\mu_c}{f_c k_c}(k_t-k),\\
0 &= \left[-(\rho+\delta_c)+(\rho+\delta_x)-\tfrac{1}{1-\beta_2}(\rho+\delta_x)\right](\mu_{ct}-\mu_c)\\
&\quad+\left[\tfrac{f_x(\rho+\delta_c)\mu_c}{f_c k_c}-\tfrac{\beta_1+\beta_2-1}{1-\beta_2}\tfrac{(\rho+\delta_x)\mu_c}{k_x}\right](k_{xt}-k_x)+\tfrac{(\rho+\delta_c)\mu_c}{f_c k_c}(k_t-k)
\end{aligned}$$

where $k_t \equiv f_c k_c + f_x k_x$. Rearranging gives:

$$\begin{aligned}
\dot{k}_t &= \tfrac{\beta_2}{1-\beta_2}\tfrac{\rho+\delta_x}{b}\tfrac{f_x k_x}{\mu_c}(\mu_{ct}-\mu_c)+\left[\tfrac{\beta_1}{1-\beta_2}\tfrac{\rho+\delta_x}{b}-(\delta_x-\delta_c)\right]f_x(k_{xt}-k_x)-\delta_c(k_t-k),\\
\dot{\mu}_{ct} &= (\rho+\delta_c)(\mu_{ct}-\mu_c)-\tfrac{(\rho+\delta_c)[\rho+(1-b)\delta_x]}{\delta_x b}\tfrac{\mu_c}{k_x}(k_{xt}-k_x)+\tfrac{(\rho+\delta_x)\delta_c b}{\rho+(1-b)\delta_x}\tfrac{\mu_c}{f_x k_x}(k_t-k),\\
0 &= -\left[(\rho+\delta_c)+\tfrac{\beta_2}{1-\beta_2}(\rho+\delta_x)\right](\mu_{ct}-\mu_c)\\
&\quad+\left[\tfrac{f_x(\rho+\delta_c)\delta_c b}{\rho+(1-b)\delta_x}-\tfrac{\beta_1+\beta_2-1}{1-\beta_2}(\rho+\delta_x)\right]\tfrac{\mu_c}{k_x}(k_{xt}-k_x)+\tfrac{(\rho+\delta_c)\delta_c b}{\rho+(1-b)\delta_x}\tfrac{\mu_c}{f_x k_x}(k_t-k).
\end{aligned}$$

The last equation can be solved for $k_{xt}-k_x$

$$k_{xt}-k_x=\frac{k_x}{\mu_c}\frac{[(1-\beta_2)(\rho+\delta_c)+\beta_2(\rho+\delta_x)](\mu_{ct}-\mu_c)+(1-\beta_2)\frac{\delta_c b(\rho+\delta_c)}{\rho+(1-b)\delta_x}\frac{\mu_c}{f_x k_x}(k_t-k)}{(1-\beta_2)\frac{b\delta_c(\rho+\delta_c)}{\rho+(1-b)\delta_x}-(\rho+\delta_x)(\beta_1+\beta_2-1)},$$

Substituting this back to the two dynamic equations leads to

$$\begin{bmatrix}\dot{k}_t\\ \dot{\mu}_{ct}\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}k_t-k\\ \mu_{ct}-\mu_c\end{bmatrix}, \tag{B.1}$$

where

$$a_{11} = \frac{\frac{(\rho+\delta_c)(\beta_1(\rho+\delta_x)-(1-\beta_2)b\delta_x)}{\rho+(1-b)\delta_x}+(\rho+\delta_x)(\beta_1+\beta_2-1)}{(1-\beta_2)\frac{b\delta_c(\rho+\delta_c)}{\rho+(1-b)\delta_x}-(\rho+\delta_x)(\beta_1+\beta_2-1)}, \tag{B.2a}$$

$$a_{12} = \left\{\frac{\beta_2}{1-\beta_2}\frac{\rho+\delta_x}{b} + \frac{\left[\frac{\beta_1}{1-\beta_2}\frac{\rho+\delta_x}{b}+(\delta_c-\delta_x)\right][(\rho+\delta_c)-\beta_2(\delta_c-\delta_x)]}{(1-\beta_2)\frac{b\delta_c(\rho+\delta_c)}{\rho+(1-b)\delta_x}-(\rho+\delta_x)(\beta_1+\beta_2-1)}\right\}\frac{f_x k_x}{\mu_c}, \tag{B.2b}$$

$$a_{21} = -\frac{(\beta_1+\beta_2-1)b(\rho+\delta_c)\frac{\rho+\delta_x}{\rho+(1-b)\delta_x}}{(1-\beta_2)\frac{b\delta_c(\rho+\delta_c)}{\rho+(1-b)\delta_x}-(\rho+\delta_x)(\beta_1+\beta_2-1)}\frac{\mu_c}{f_x k_x}, \tag{B.2c}$$

$$a_{22} = -\frac{(\rho+\delta_c)\left[\frac{b\delta_c(\rho+\delta_x)}{\rho+(1-b)\delta_x}+(\rho+\delta_x)(\beta_1+\beta_2-1)\right]}{(1-\beta_2)\frac{b\delta_c(\rho+\delta_c)}{\rho+(1-b)\delta_x}-(\rho+\delta_x)(\beta_1+\beta_2-1)}. \tag{B.2d}$$

The determinant and the trace of the matrix in (B.1) are found to be:

$$\text{Det} = \frac{\delta_c(\rho+\delta_c)[\rho+(1-b)\delta_x](1-\beta_1)}{(\rho+\delta_x)[\rho+(1-b)\delta_x](\beta_1+\beta_2-1)-b\delta_c(\delta_c+\rho)(1-\beta_2)}, \tag{B.3a}$$

$$\text{Tr} = \frac{\rho(\rho+\delta_x)[\rho+(1-b)\delta_x](\beta_1+\beta_2-1)+\delta_c(\delta_c+\rho)[b(\delta_x+\rho\beta_2)-\beta_1(\rho+\delta_x)]}{(\rho+\delta_x)[\rho+(1-b)\delta_x](\beta_1+\beta_2-1)-b\delta_c(\delta_c+\rho)(1-\beta_2)}. \tag{B.3b}$$

B.1.2 Characterization of the stability properties

The steady state is saddle–path stable if $\text{Det} < 0$, it is stable if $\text{Tr} < 0 < \text{Det}$, and it is unstable if $\text{Tr}, \text{Det} > 0$. In order to characterize the different cases, first note that the denominators of the trace and the determinant are the same. Second, the numerator of the determinant is always positive. So the local stability properties will depend only on the signs of the numerator of the trace and on the common denominator. Through β_1 and β_2 they both depend on θ_x, so we will write $N(\theta_x)$ and $D(\theta_x)$. To find their signs, we first find the values of θ_x for which they become zero:

$$D(\underline{\theta}_x) = 0 \iff \underline{\theta}_x = \frac{b^2\delta_c(\rho+\delta_c)}{(\rho+\delta_x)[\rho+(1-b)\delta_x]+(1-b)b\delta_c(\rho+\delta_c)} \tag{B.4a}$$

$$N(\overline{\theta}_x) = 0 \iff \overline{\theta}_x = \frac{b^2\delta_c(\rho+\delta_c)}{(\rho+\delta_x)[\rho+(1-b)\delta_x]-\frac{\rho b+\delta_x}{\rho}b\delta_c(\rho+\delta_c)} \tag{B.4b}$$

We can see that $D(\theta_x) < 0$ if and only if $\theta_x < \underline{\theta}_x$, $D(\theta_x) > 0$ if and only if $\theta_x > \underline{\theta}_x$, $N(\theta_x) < 0$ if and only if $\theta_x < \overline{\theta}_x$, and $N(\theta_x) > 0$ if and only if $\theta_x > \overline{\theta}_x$. Now, if the condition in (i.a) holds then $0 < \underline{\theta}_x < \overline{\theta}_x$ and if the condition in (i.b) holds then $\underline{\theta}_x < 0 < \overline{\theta}_x$. Using this to determine the signs of the determinant and the trace proves our claims.

B.2 Strictly quasi–convex *f*

B.2.1 Computation of the determinant and the trace

We again represent the steady values of f, g, and their derivatives by dropping their arguments, so $f \equiv f\left(\frac{x_c}{x_x}, 1\right)$, $g \equiv g\left(\frac{x_c}{x_x}\right)$, etc. We start the proof by listing some helpful identities that have to hold in our model. First, the definition of g as the inverse of $\frac{f_c}{f_x}$ implies that

$$g' = \frac{f_x^2}{f_{cc}f_x - f_c f_{xc}}. \tag{B.5a}$$

Second, the linear homogeneity of f implies:

$$f = gf_c + f_x, \qquad 0 = gf_{cc} + f_{cx}, \qquad 0 = f_{xx} + gf_{cx}. \tag{B.5b}$$

Third, (A.6) and (B.5b) give

$$\frac{\rho+\delta_x(1-b)}{b\delta_x} = \frac{gf_c}{f_x}, \qquad \frac{\rho+\delta_x(1-b)}{\rho+\delta_x} = \frac{gf_c}{f}, \tag{B.6a}$$

Finally, using this and (B.5a), we find:

$$\frac{f_{xc}}{f_x} g' \frac{\mu_c}{\mu_x} = \frac{f_{xc} f_c}{f_{cc} f_x - f_c f_{xc}} = -\frac{g f_c}{f_x + g f_c} = -\frac{\rho+\delta_x(1-b)}{\rho+\delta_x} \tag{B.6b}$$

The first step of the derivation of the determinant and the trace is to linearize the reduced-form dynamics at the steady state. Indicating steady state variables by dropping the time subscript, the result is:

$$\begin{bmatrix} \dot{k}_{ct} \\ \dot{k}_{xt} \\ \dot{\mu}_{ct} \\ \dot{\mu}_{xt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} k_{ct} - k_c \\ k_{xt} - k_x \\ \mu_{ct} - \mu_c \\ \mu_{xt} - \mu_x \end{bmatrix}, \tag{B.7}$$

where:[18]

$$a_{11} = -\delta_c, \qquad a_{12} = \frac{\beta_1}{1-\beta_2} \frac{\delta_c k_c}{k_x}, \qquad a_{13} = \left[\frac{g'}{g} \frac{\mu_c}{\mu_x} - \frac{\beta_2}{1-\beta_2} \frac{f_{xc}}{f_x} g' \frac{\mu_c}{\mu_x} - \frac{g f_c}{f} \frac{g'}{g} \frac{\mu_c}{\mu_x} \right] \frac{\delta_c k_c}{\mu_c},$$

$$a_{14} = \left[\frac{\beta_2}{1-\beta_2} - \frac{g'}{g} \frac{\mu_c}{\mu_x} + \frac{\beta_2}{1-\beta_2} \frac{f_{xc}}{f_x} g' \frac{\mu_c}{\mu_x} + \frac{f_c}{f} g' \frac{\mu_c}{\mu_x} \right] \frac{\delta_c k_c}{\mu_x}, \qquad a_{21} = 0, \qquad a_{22} = \frac{\beta_1}{1-\beta_2} \frac{\delta_x k_x}{k_x} - \delta_x,$$

$$a_{23} = \left[\frac{\beta_2}{1-\beta_2} \frac{f_{xc}}{f_x} g' \frac{\mu_c}{\mu_x} - \frac{f_c}{f} g' \frac{\mu_c}{\mu_x} \right] \frac{\delta_x k_x}{\mu_c}, \qquad a_{24} = \left[\frac{\beta_2}{1-\beta_2} - \frac{\beta_2}{1-\beta_2} \frac{f_{xc}}{f_x} g' \frac{\mu_c}{\mu_x} + \frac{g f_c}{f} \frac{g'}{g} \frac{\mu_c}{\mu_x} \right] \frac{\delta_x k_x}{\mu_x},$$

$$a_{31} = \frac{(\rho+\delta_c)\mu_c}{k_c}, \qquad a_{32} = 0, \qquad a_{33} = \rho + \delta_c, \qquad a_{34} = 0,$$

$$a_{41} = 0, \qquad a_{42} = \frac{\beta_1+\beta_2-1}{1-\beta_2} \frac{(\rho+\delta_x)\mu_x}{k_x}, \qquad a_{43} = \frac{1}{1-\beta_2} \frac{f_{xc}}{f_x} g' \frac{\mu_c}{\mu_x} \frac{(\rho+\delta_x)\mu_x}{\mu_c},$$

$$a_{44} = (\rho + \delta_x) - \frac{1}{1-\beta_2} (\rho + \delta_x) - \frac{1}{1-\beta_2} (\rho + \delta_x) \frac{f_{xc}}{f_x} g' \frac{\mu_c}{\mu_x}.$$

To simplify these expressions, it is useful to define the elasticity of the investment ratio with respect to the relative price evaluated at the steady state. Denoting the inverse of that elasticity by $\varepsilon \geq 0$, we have:

$$\varepsilon \equiv \frac{g\left(\frac{\mu_c}{\mu_x},1\right)}{g'\left(\frac{\mu_c}{\mu_x},1\right)} \frac{1}{\frac{\mu_c}{\mu_x}}. \tag{B.8}$$

[18]To find these expressions we have repeatedly used the fact that if a function is of the form $h(x_1, x_2, x_3) = x_1^\alpha x_2^\beta - a x_3$, then its partial derivative can be written as $\frac{\partial h}{\partial x_1} = \alpha \frac{f(x_1,x_2,x_3)+a x_3}{x_1}$.

Now, using (B.6a) and (B.6b), the previous terms can be rewritten:

$$a_{11} = -\delta_c, \quad a_{12} = \frac{\beta_1}{1-\beta_2}\frac{\delta_c k_c}{k_x}, \quad a_{13} = \left[\frac{\beta_2}{1-\beta_2} + \frac{1}{\varepsilon}\frac{1-(1+\varepsilon)\beta_2}{1-\beta_2}\frac{\delta_x b}{\rho+\delta_x}\right]\frac{\delta_c k_c}{\mu_c}, \tag{B.9a}$$

$$a_{14} = -\frac{1}{\varepsilon}\frac{1-(1+\varepsilon)\beta_2}{1-\beta_2}\frac{\delta_c k_c}{\mu_x}\frac{\delta_x b}{\rho+\delta_x}, \quad a_{21} = 0, \quad a_{22} = \delta_x\frac{\beta_1+\beta_2-1}{1-\beta_2}, \tag{B.9b}$$

$$a_{23} = -\frac{1}{\varepsilon}\frac{1-(1+\varepsilon)\beta_2}{1-\beta_2}\frac{\delta_x k_x}{\mu_c}\frac{\rho+\delta_x(1-b)}{\rho+\delta_x}, \quad a_{24} = \left[\frac{\beta_2}{1-\beta_2} + \frac{1}{\varepsilon}\frac{1-(1+\varepsilon)\beta_2}{1-\beta_2}\frac{\rho+\delta_x(1-b)}{\rho+\delta_x}\right]\frac{\delta_x k_x}{\mu_x}, \tag{B.9c}$$

$$a_{31} = \frac{(\rho+\delta_c)\mu_c}{k_c}, \quad a_{32} = 0, \quad a_{33} = \rho+\delta_c, \quad a_{34} = a_{41} = 0, \tag{B.9d}$$

$$a_{42} = -\frac{\beta_1+\beta_2-1}{1-\beta_2}\frac{\mu_x}{k_x}(\rho+\delta_x), \quad a_{43} = -\frac{1}{1-\beta_2}[\rho+\delta_x(1-b)]\frac{\mu_x}{\mu_c}, \quad a_{44} = (\rho+\delta_x) - \frac{1}{1-\beta_2}\delta_x b. \tag{B.9e}$$

The second step is to combined the terms just derived and actually compute the determinant and the trace. Using the fact that $a_{32} = a_{34} = a_{41} = 0$, the determinant can be written as

$$\begin{aligned}\text{Det} &= a_{31}a_{42}(a_{13}a_{24} - a_{14}a_{23}) + a_{22}a_{31}(a_{14}a_{43} - a_{13}a_{44})\\ &\quad + a_{11}a_{33}(a_{22}a_{44} - a_{24}a_{42}) + a_{12}a_{31}(a_{23}a_{44} - a_{24}a_{43}).\end{aligned}$$

Using the previous expressions, the four terms in that determinant are found to equal:

$$\begin{aligned}a_{31}a_{42}(a_{13}a_{24} - a_{14}a_{23}) &= -\frac{1}{\varepsilon}\frac{\beta_2}{1-\beta_2}\frac{\beta_1+\beta_2-1}{1-\beta_2}\delta_c\delta_x(\rho+\delta_c)(\rho+\delta_x),\\ a_{22}a_{31}(a_{14}a_{43} - a_{13}a_{44}) &= \frac{\beta_1+\beta_2-1}{1-\beta_2}\frac{\beta_2}{1-\beta_2}\frac{(1+\varepsilon)\delta_x b-\varepsilon(\rho+\delta_x)}{\varepsilon}\delta_c\delta_x(\rho+\delta_c),\\ a_{11}a_{33}(a_{22}a_{44} - a_{24}a_{42}) &= -\frac{\beta_1+\beta_2-1}{1-\beta_2}\frac{1+\varepsilon}{\varepsilon}\delta_x\delta_c(\rho+\delta_c)[\rho+\delta_x(1-b)],\\ a_{12}a_{31}(a_{23}a_{44} - a_{24}a_{43}) &= \frac{\beta_1}{1-\beta_2}\frac{\beta_2}{1-\beta_2}\frac{1+\varepsilon}{\varepsilon}\delta_c\delta_x(\rho+\delta_c)[\rho+\delta_x(1-b)].\end{aligned}$$

Using these expressions and simplifying, we find the determinant:

$$\text{Det} = \frac{1+\varepsilon}{\varepsilon}\frac{\delta_c\delta_x(\rho+\delta_c)[\rho+\delta_x(1-b)](1-\beta_1)}{1-\beta_2}. \tag{B.10}$$

In general form the trace is given by:

$$\mathrm{Tr} = a_{11} + a_{22} + a_{33} + a_{44}.$$

Substituting in the previous expressions for a_{ii}, we find the trace:

$$\mathrm{Tr} = 2\rho + \delta_x \tfrac{\beta_1 - b}{1-\beta_2}. \tag{B.11}$$

B.2.2 Characterization of the stability properties

We start with the case $\theta_x \in [0, \frac{b}{1-b})$, implying that $\beta_2 < 1$. Then Det > 0 and Tr > 0.[19] Now suppose that the steady state were stable. Then (B.7) would have three or four eigenvalues with negative real parts. If (B.7) had four eigenvalues with negative real parts, then the trace would have to be negative, which is a contraction. If (B.7) had three eigenvalues with negative real part, then the determinant would have to be negative, which is a contradiction.

We continue with the case $\theta_x \in [\frac{b}{1-b}, \frac{1-b}{b})$, implying that $\beta_2 > 1$. Then Det < 0. Suppose that the steady state were saddle–path stable. Then (B.7) would have two eigenvalues with negative real part and two eigenvalues with positive real part. Irrespective of whether they are real or complex conjugates, this would imply that the determinant must become positive, which is a contraction.

C Proof of Proposition 3

The proof of this proposition follows because using $\frac{\mu_x}{\mu_c} = \frac{f_x}{f_c}$, one can show that the limits of the steady state versions of the four equations in (A.4), which characterize uniquely the steady state with quasi-convex f, imply the three equations in (A.11), which characterize uniquely the steady state with linear f. In particular, f_c times (A.4c) plus f_x times (A.4d) converges to

[19] Recall that $\beta_1 = (1 + \theta_x)b$, so $\beta_1 - b = \theta_x b \geq 0$.

(A.11a). Second, (A.4a) is identical to equation (A.11b). Third, f_c times (A.4b) minus f_x times (A.4a) converges to (A.11c).

D Proof of Proposition 4

We start by defining the production possibility frontier between the consumption good, c_t, and the composite investment good, $x_t \equiv f(x_{ct}, x_{xt})$:

$$\max_{x_{ct-\Delta t}, x_{xt-\Delta t}, l_{xt}, l_{ct}} x_t(c_t) \tag{D.1a}$$

$$\text{s.t.} \quad x_t \leq k_{xt}^{\beta_1} l_{xt}^{\beta_2}, \qquad c_t \leq k_{ct}^{\alpha_1} l_{ct}^{\alpha_2}, \qquad l_{ct} + l_{xt} \leq \bar{l}_t, \tag{D.1b}$$

$$k_{ct} \leq (x_{ct-\Delta t} - \delta_c \bar{k}_{ct-\Delta t})\Delta t + \bar{k}_{ct-\Delta t}, \qquad k_{xt} \leq (x_{xt-\Delta t} - \Delta_x \bar{k}_{xt-\Delta t})\Delta t + \bar{k}_{xt-\Delta t}, \tag{D.1c}$$

$$f(x_{ct-\Delta t}, x_{xt-\Delta t}) \leq \bar{x}_{t-\Delta t}, \tag{D.1d}$$

where $\bar{l}_t, \bar{k}_{ct-\Delta t}, \bar{k}_{xt-\Delta t}, \bar{x}_{t-\Delta t}$ are given. The solution to this problem determines for given feasible c_t the maximal level of x_t. We use Δt in writing this problem because of the sector–specificity of capital, which means that at some time $t - \Delta t$, Δ being small, the two new capital goods need to be chosen.

Now rewrite the problem as:

$$\max_{x_{ct-\Delta t}, x_{xt-\Delta t}, l_{xt}} \left[(x_{xt-\Delta t} - \delta_x \bar{k}_{xt-\Delta t})\Delta t + \bar{k}_{xt-\Delta t}\right]^{\beta_1} l_{xt}^{\beta_2} \tag{D.2a}$$

$$\text{s.t.} \qquad c_t = \left[(x_{ct-\Delta t} - \delta_c \bar{k}_{ct-\Delta t})\Delta t + \bar{k}_{ct-\Delta t}\right]^{\alpha_1} \left[\bar{l}_t - l_{xt}\right]^{\alpha_2}, \tag{D.2b}$$

$$\bar{x}_{t-\Delta t} = f(x_{ct-\Delta t}, x_{xt-\Delta t}). \tag{D.2c}$$

The necessary first-order conditions are:

$$x_t = \left[x_{xt-\Delta t} - \delta_x \bar{k}_{xt-\Delta t})\Delta t + \bar{k}_{xt-\Delta t}\right]^{\beta_1} l_{xt-\Delta t}^{\beta_2} \tag{D.3a}$$

$$c_t = \left[(x_{ct-\Delta t} - \delta_c \bar{k}_{ct-\Delta t})\Delta t + \bar{k}_{ct-\Delta t}\right]^{\alpha_1} \left[\bar{l}_t - l_{xt}\right]^{\alpha_2} \tag{D.3b}$$

$$\frac{b}{1-b}\frac{l_{xt}}{(x_{xt-\Delta t}-\delta_x k_{xt-\Delta t})\Delta t+k_{xt-\Delta t}}\frac{f_c(x_{ct-\Delta t},x_{xt-\Delta t})}{f_x(x_{ct-\Delta t},x_{xt-\Delta t})} = \frac{a}{1-a}\frac{\bar{l}-l_{xt}}{(x_{ct-\Delta t}-\delta_c k_{ct-\Delta t})\Delta t+k_{ct-\Delta t}}, \tag{D.3c}$$

$$\bar{x}_{t-\Delta t} = f(x_{ct-\Delta t}, x_{xt-\Delta t}). \tag{D.3d}$$

These four equations define the production possibility frontier between c_t and x_t.

Inspecting the optimization problem in (D.2) for $f(x_{ct}, x_{xt}) = f_c x_{ct} + f_x x_{xt}$, we can see that the constraint (D.2c) becomes

$$\bar{x}_{t-\Delta t} = f_c x_{ct-\Delta t} + f_x x_{xt-\Delta t}. \tag{D.2c'}$$

Therefore the first-order conditions are (D.3a) and (D.3b) as before and

$$\frac{b}{1-b}\frac{l_{xt+\Delta t}}{(x_{xt-\Delta t}-\delta_x k_{xt-\Delta t})\Delta t+k_{xt-\Delta t}}\frac{f_c}{f_x} = \frac{a}{1-a}\frac{\bar{l}_{t+\Delta t}-l_{xt+\Delta t}}{(x_{ct-\Delta t}-\delta_c k_{ct-\Delta t})\Delta t+k_{ct-\Delta t}}. \tag{D.3c'}$$

$$f_c x_{ct-\Delta t} + f_x x_{xt-\Delta t} = \bar{x}_{t-\Delta t}. \tag{D.3d'}$$

(D.3c) and (D.3d) converge to (D.3c') and (D.3d') as $f_i \to f$ in $U(x_c, x_x)$.

References

Aiyagari, S. Rao, "Comments on Farmer and Guo's "The Econometrics of Indeterminacy: An Applied Study"," *Carnegie Rochester Conference Series on Public Policy*, 1995, *43.*

Bartelsman, Eric, Ricardo Caballero, and Richard K. Lyons, "Consumer and Supplier Driven Externalities," *American Economic Review*, 1994, *84*, 1075–1084.

Basu, Susanto and John G. Fernald, "Returns to Scale in U.S. Production: Estimates and Implications," *Journal of Political Economy*, 1997, *105*, 249–283.

Benhabib, Jess and Roger E. A. Farmer, "Indeterminacy and Increasing Returns," *Journal of Economic Theory*, 1994, *63*, 19–41.

——— **and** ———, “Indeterminacy and Sector-Specific Externalities,” *Journal of Monetary Economics*, 1996, *37*, 421–443.

——— **and** ———, “Indeterminacy and Sunspots in Macroeconomics,” in John B. Taylor and Michael Woodford, eds., *Handbook of Macroeconomics*, Amsterdam: North-Holland, 1999.

———**, Qinglai Meng, and Kazuo Nishimura**, “Indeterminacy under Constant Returns to Scale in Multisector Economies,” *Econometrica*, 2000, *68*, 1541–1548.

Boldrin, Michele and Aldo Rustichini, “Growth and Indeterminacy in Dynamic Models with Externalities,” *Econometrica*, 1994, *62*, 323–342.

———**, Lawrence J. Christiano, and Jonas D.M. Fisher**, “Habit Persistence, Asset Returns and the Business Cycle,” *American Economic Review*, 2001, *91*, 149–166.

Burnside, Craig, Martin Eichenbaum, and Sergio Rebelo, “Capital Utilization and Returns to Scale,” in Ben S. Bernanke and Julio J. Rotemberg, eds., *NBER Macroeconomics Annual 1995*, Cambridge, MA: MIT Press, 1995.

Christiano, Lawrence J., “A Discrete-Time Version of Benhabib-Farmer II,” Manuscript, Northwestern University, Evanston, IL 1995.

Farmer, Roger E. A. and Jang Ting Guo, “Real Business Cycles and the Animal Spirit Hypothesis,” *Journal of Economic Theory*, 1994, *63*, 42–73.

Fisher, Jonas D. M., “Relative Prices, Complementarities, and Comovement among Components of Aggregate Expenditure,” *Journal of Monetary Economics*, 1997, *39*, 449–474.

Guo, Jang-Ting and Kevin J. Lansing, “Fiscal Policy, Increasing Returns and Endogenous Fluctuations,” *forthcoming: Macroeconomic Dynamics*, 2001.

Hall, Robert E., “Relation Between Price and Marginal Cost in U.S. Industry,” *Journal of Political Economy*, 1988, *96*, 921–947.

Hammermesh, Daniel S. and Gerard Pfann, "Adjustment Costs and Factor Demand," *Journal of Economic Literature*, 1996, *39*, 1264–1292.

Hansen, Gary D., "Indivisible Labor and the Business Cycle," *Journal of Monetary Economics*, 1985, *16*, 309–328.

Harrison, Sharon G. and Mark Weder, "Tracing Externalities as Sources of Indeterminacy," *Journal of Economic Dynamics and Control*, 2001, *26*, 851–867.

Herrendorf, Berthold, Ákos Valentinyi, and Robert Waldmann, "Ruling Out Multiplicity and Indeterminacy: The Role of Heterogeneity," *Review of Economic Studies*, 2000, *67*, 295–307.

——— **and** ———, "Determinacy Through Intertemporal Capital Adjustment Costs," Manuscript, Universidad Carlos III de Madrid 2002.

Huffman, Gregory W. and Mark A. Wynne, "The Role of Intratemporal Adjustment Costs in a Multisector Economy," *Journal of Monetary Economics*, 1999, *43*, 317–350.

Kim, Jinill, "Indeterminacy and Investment Adjustment Cost," Discussion Paper 38, Federal Reserve Board, Washington, D.C. 1998.

King, Robert G., Charles I. Plosser, and Sérgio T. Rebelo, "Production, Growth and Business Cycles: I. The Basic Neoclassical Model," *Journal of Monetary Economics*, 1988, *21*, 195–232.

Lucas, Robert E. Jr. and Edward C. Prescott, "Investment under Uncertainty," *Econometrica*, 1971, *39*, 659–681.

Morris, Stephen and Hyun Song Shin, "Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks," *American Economic Review*, 1998, *88*, 587–597.

Perli, Roberto, "Indeterminacy, Home Production, and the Business Cycle: A Calibrated Analysis," *Journal of Monetary Economics*, 1998, *41*, 105–125.

Ramey, Valerie A. and Matthew D. Shapiro, "Displaced Capital: A Study of Aerospace Plant Closings," *Journal of Political Economy*, 2001, *109*, 958–992.

Schmitt-Grohe, Stephanie, "Endogenous Business Cycles and the Dynamics of Output, Hours, and Consumption," *American Economic Review*, 2000, *90*, 1136–1159.

Shannon, Chris and William R. Zame, "Quadratic Concavity and Determinacy of Equilibrium," *Econometrica*, 2002, *70*, 631–662.

Weder, Mark, "Fickle Consumers, Durable Goods, and Business Cycles," *Journal of Economic Theory*, 1998, *81*, 37–57.

Wen, Yi, "Indeterminacy, Dynamic Adjustment Costs, and Cycles," *Economics Letters*, 1998, *59*, 213–216.