

A Detail-free Mediator

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Abstract

Two players are allowed to communicate repeatedly through a mediator and then have direct communication. The device receives private inputs from each of the two players and produces as a public output a fixed deterministic function of the inputs. After the communication terminates, the players play an arbitrary finite complete information normal form game. We show that any correlated equilibrium of the original game can be approached as ϵ -Nash equilibria of the game extended with the communication phases. In particular, the mediator is the same for all games and for all equilibrium distributions.

Keywords: communication device, correlated equilibrium, repeated communication, detail-free mechanism

1 Introduction

The concept of correlated equilibrium (CE)(Aumann 1974) assumes a mediator who randomizes over the action profiles and then sends private messages to the players. These private messages can be interpreted as suggested actions for the players. The players follow the suggestion if it is a best play given their information about the other player's action. By the help of a such mediator the players can coordinate their actions. Moreover,

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such an extension of a game enlarges the set of equilibrium payoffs. The problem is that the mediator has to be tailor-made to the game and distribution at hand.

A plausible question is whether there exists a *common* mediator who can help the players to implement any CE of any game. This is the idea of Wilson's (1987) "detail-free" mechanism¹.

Our answer to the question is positive. We suggest the following protocol. Two players are allowed to communicate repeatedly through the mediator *AND*. The *AND* device receives as inputs 0 or 1 from each of the two players, and produces as an output 1 if both inputs were 1, and 0 otherwise.

<i>AND</i>	0	1
0	0	0
1	0	1

After the *mediated* communication terminates, the players engaged in a *direct* repeated communication phase where the messages are sent simultaneously. Finally the players play a finite complete information normal form game. We show that any correlated equilibria of the one-shot game can be approached as ϵ -Nash equilibria of the game extended by the mediated and direct communication phase.

Several articles investigate how to circumvent the necessity of the mediator or, at least, in some sense minimize his duties as the game or the implemented CE varies. An important result is that two players can solve the communication problem without any mediation if they are computationally restricted (Urbano, Vila 2002), otherwise a third party is needed.

Lehrer (1996), Lehrer and Sorin (1997) and Vida (2003) show how the players can replace fortune and avoid the use of private messages from the mediator. That is, the mediator's task is to make public announcements, which are deterministic functions of the players' private messages. However, in all solutions proposed in these papers, the mediator has to be tailor-made to the particular game or CE at hand.

Lehrer (1991) studies the *AND* mechanism in the context of infinitely repeated games with imperfect monitoring. He shows that any correlated distribution can be generated by a jointly controlled correlation phase. Deviations in this phase can be statistically detected² and punished immediately. However, in the case of a one-shot game extended by repeated

¹Thanks to Andrew Postlewaite for pointing out this connection with the literature.

²By the help of the reporting phase otherwise undetectable deviations can be ruled out.

communication through the *AND* no correlation can be achieved securely³(Gossner and Vieille 2001).

In this paper we find that direct communication after the repeated mediated communication phase resolves the sharp contrast between the two results above⁴. We apply Lehrer's (1991) protocol and show that by repetition it becomes incentive compatible even in the case of one-shot games. The main idea is that by direct communication the players can jointly choose one of the instances of the jointly controlled correlations. Then, the players reveal all their past messages except those corresponding to the chosen correlation phase. The choosing and the reporting phase⁵ have to be done through simultaneous direct communication. This is why direct talk is essential to achieve incentive compatibility.

Our conjecture is that the same idea leads to "universal" implementation of the set of communication equilibria (Forges 1986, 1990) in case of games with incomplete information with two or three players (see the corresponding result of Ben-Porath (2003)).

The paper is structured as follows. Section 2 shows an easy example in the case of the chicken game and introduces some terminology. In section 3 we go to the general notation and concepts and we state the result. Section 4 contains the proof accompanied by an elaborated example. Finally we conclude.

2 Example

We take a simple 2×2 game and one of its correlated equilibria. First we show how Lehrer's protocol generates the desired distribution when the players do not deviate from the prescribed randomization. Second we stress that players have incentive to deviate and by manipulating the protocol induce a distribution that is more favorable for them. Then we show that by repeating the procedure such deviations are going to be detectable statistically and could be punished. Another problem arises by the repetition of the protocol. Namely the players cannot coordinate on which instance of the repetition they should play. Obviously, none of them can suggest one of the instances nor can they agree in advance. The problem is solved by the direct communication phase. This gives the players the

³Secureness is a more demanding property of a protocol than maintaining equilibrium (see Gossner(1998))

⁴However our protocol is still not secure. Our incentive compatibility condition is ϵ -Nash equilibrium of the extended game

⁵Introduced by Lehrer (1991).

possibility to jointly choose a stage of the mediated phase and coordinate their actions accordingly. Because of the simple structure of the distribution there is no need for the reporting phase⁶. Consider the chicken game⁷:

A	0	1
0	6,6	2,7
1	7,2	0,0

A correlated equilibrium distribution of the game:

μ	0	1
0	$\frac{1}{2}$	$\frac{1}{4}$
1	$\frac{1}{4}$	0

The players send their "intended actions" privately to the mediator, who notifies them (announcing publicly 1) in case the intended action-profile was (1, 1). That is by using the *AND* he computes the public signal.

<i>AND</i>	0	1
0	0	0
1	0	1

Notice that whenever a player sends 1, she will be able to infer the message what the other player has sent. On the other hand if she sends 0 she gets no information about the other's private message.

Let the players send messages until the first 0 public announcement is made. We say that this *communication round* of length $p = 1$ was *successful*. Consider the communication strategies such that the players are randomizing between 0 and 1 with probability $(\frac{2}{3}, \frac{1}{3})$. The induced distribution on the message profiles is:

P	0	1
0	$\frac{4}{9}$	$\frac{2}{9}$
1	$\frac{2}{9}$	$\frac{1}{9}$

Hence conditional on the event that the public announcement is not 1, the believed probabilities over the sent message-profiles are exactly:

$P(. 0)$	0	1
0	$\frac{1}{2}$	$\frac{1}{4}$
1	$\frac{1}{4}$	0

⁶The example in the proof can show the necessity of such a phase.

⁷The game is due to Aumann and the example is due to Lehrer.

So, the players could peg their strategies to their information available and play the correlated equilibrium μ by simply playing as their intended actions were at that round.

Obviously, this protocol is manipulable, since for example the row player has incentives to send 1 always, and play his preferred equilibrium (1,0). Notice that doing so the induced theoretical distribution on the message-profiles is changed to:

P	0	1
0	0	0
1	$\frac{2}{3}$	$\frac{1}{3}$

To make the protocol incentive compatible we need to make these deviations detectable. So we suggest the following. The players repeat the above process, that is randomize again with $(\frac{2}{3}, \frac{1}{3})$ until another successful round occurs. This is going to be the second successful round. The exact time of the first and second successful rounds are random variables (κ_1, κ_2) . Now let players repeat the mediated communication until $T = 16$ successful communication round has occurred.

Now the players can perform a standard statistical test about the randomization of the opponent. This is due to sending the message 1 reveals the other's message. Put it simply, the ratio of the public announcements 1 has to be close to $\frac{1}{9}$. If T is big enough any deviation in the above randomization is statistically detectable.

The players should coordinate on one of the successful rounds κ^* and play accordingly. After the mediated communication phase letting the players communicate directly allows them to jointly select one of the successful rounds. This can be done by a simple jointly controlled lottery on $\kappa_1, \dots, \kappa_T$. By mixing their messages with probabilities (0.5, 0.5) for $O = \log T = \log 16 = 4$ times repeatedly. If the resulting direct communication sequence was:

$$(0, 0), (1, 0), (1, 1), (0, 1)$$

the "interpretation" of these message pairs can be fixed as follows:

$$(0, 0) = 0, (1, 0) = 1, (1, 1) = 0, (0, 1) = 1$$

We define the players decision rules after the communication terminates as: "Choose the action you intended to move" in the:

$$0 * 2^0 + 1 * 2^1 + 0 * 2^2 + 1 * 2^3 = 10th$$

successful round of the mediated communication (that is the $\kappa^* = \kappa_{10}$ th round of the whole mediated communication), if no deviation was detected. Otherwise play 1, that is your punishment strategy.

Notice that the direct communication round in fact selects a successful communication round with equal probabilities in a non-manipulable way, that is selects an action profile with probability distribution close to μ .

Since the min max payoffs are always less than or equal to the correlated equilibrium payoffs no players have incentives to deviate from the prescribed communication strategy and decision rule. Moreover if no deviation took place then the players conditional beliefs about the other player action (knowing their own) coincides with that of in μ .

The method can be generalized to other CE distributions. For example if the players like to implement:

μ'	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	$\frac{1}{3}$	0

all they have to do is to change their randomization in the mediated phase to (0.5, 0.5).

More complicated distributions, for example

μ''	0	1
0	$\frac{2}{5}$	$\frac{1}{5}$
1	$\frac{1}{5}$	$\frac{1}{5}$

need more mediated stages ($p > 1$). That is a mediated round can be longer than 1. These kind of distributions also give rise other possible deviations which cannot be detected statistically. In these cases the reporting phase can make Lehrer's protocol incentive compatible (see the details in the proof).

3 Concepts and the result

This section is devoted to introduce a general notation to describe the game extended with the various communication phases. At different stages of the communication the players have different information concerning the past. Moreover, at every stage they learn some new information which depends on the message they have sent and whether the stage was

mediated or not. After fixing the timing we define the strategy space of the extended game. Finally, we are ready to state the main result.

3.1 Notations

Consider a general finite 2 player normal form game with complete information $G = (g, A)$, where $A = A^1 \times A^2$ the set of action-profiles and g^i is the payoff function for player $i \in \{1, 2\}$. We follow the notation of Gossner and Vieille (2001). $M^i = \{0, 1\}$ is the players' message space. To simplify the notation label the different input combinations as:

$h(m)$	0	1
0	a	c
1	b	$*$

The *AND* signaling function l^1 for player 1 can be described as $l^1(0, \cdot) = \{a, c\}$ and $l^1(1, 0) = \{b\}$ and $l^1(1, 1) = \{*\}$. l^1 induces $\mathcal{P}^1 = \{\{a, c\}, \{b\}, \{*\}\}$ an information partition. For player 2 symmetrically it is $\mathcal{P}^2 = \{\{a, b\}, \{c\}, \{*\}\}$. Let the players send messages $m_n = (m_n^1, m_n^2)$ simultaneously for $0 \leq n \in \mathbb{N}$. Player i is told $l^i(m_n) \in \mathcal{P}^i$.

The set of plays at time n is $H_n = \{a, b, c, *\}^n$ and denote $H_\infty = \{a, b, c, *\}^\mathbb{N}$. We denote $h(m_n)$ the play at stage n . Prior to sending the message in stage n the information available for player i is \mathcal{H}_n^i an algebra generated by the cylinder sets of the form $h_n^i \times H_\infty$, where $h_n^i \in (\mathcal{P}^i)^n$ is a sequence of n elements. Denote $\mathcal{H}_\infty^i = \sigma(\mathcal{H}_n^i, n \geq 0)$ the σ -algebras over H_∞ generated by these σ -algebras. Define \mathcal{H}_∞ similarly. Given a finite set E denote ΔE the set of probability distributions over the set E .

Fix a $p \in \mathbb{N}$ and denote for $k \geq 1$ ${}_k h = (h(m_{(k-1)p}), \dots, h(m_{kp-1})) \in H_p$, that is the k th p -coordinates of a given play h . We call ${}_k h$ the k th *communication round* of length p . Say that the *communication was successful* in the k th round, if ${}_k h$ does not contain $\{*\}$. Let $1 \leq \kappa_1 < \dots < \kappa_T$ denote the first T successful rounds.

Assume that after $\kappa_T p$ stages of the mediated communication the players communicate directly. Let $t \geq 0$ denote the t th stage of the direct communication. The information partitions corresponding to the trivial signalling functions of the direct communication are $\mathcal{P}_d^i = \{\{a\}, \{b\}, \{c\}, \{*\}\}$ for $i = 1, 2$. The information available for player i at stage $\kappa_T p + t$ can be described by the σ -algebra $\mathcal{H}_{\kappa_T p}^i \otimes 2^{H_t}$.

3.2 The extended game

Now we fix the timing of the extended game and define its strategy-space.

Consider the following timing:

1. the players communicate under the mechanism *AND* until T successful communication round of length p occurs⁸,
2. at stage $\kappa_T p$ start communicate directly and simultaneously $O + \kappa_T p$ times repeatedly
3. finally choose an action.

PICTURE FOR THE PROXIES

Denote the extended game with $\Gamma_p(T)$. The strategy space of the extended game is:

1. the communication strategies through the *AND*: $\sigma^i = (\sigma_n^i)_{n \geq 0}$, where σ_n^i is \mathcal{H}_n^i -measurable mapping to ΔM^i .
2. the direct communication strategies: $\tau^i = (\tau_t^i)_{O + \kappa_T p > t \geq 0}$, where τ_t^i is $\mathcal{H}_{\kappa_T p}^i \otimes 2^{H_t}$ -measurable mapping to ΔM^i
3. decision rules ρ^i $\mathcal{H}_{\kappa_T p}^i \otimes 2^{H_{O + \kappa_T p}}$ -measurable mapping to ΔA^i and ρ_∞^i \mathcal{H}_∞^i -measurable if there were no T successful rounds.

Denote $\pi = (\sigma, \tau, \rho)$ a strategy profile of the extended game. There is an induced distribution P_π on $(H_\infty \times H_\infty \times A, \mathcal{H}_\infty \otimes \mathcal{H}_\infty \otimes 2^A)$.

3.3 The result

We are ready to state the main result.

Definition 1 π is ϵ -Nash equilibrium of the extended game, if for any i and π^i

$$\mathbb{E}_{P_\pi} g^i(a) + \epsilon \geq \mathbb{E}_{P_{\pi^i, \pi^{-i}}} g^i(a).$$

Definition 2 An information structure on a set A is a probability distribution μ over A . An element $a = (a_1, a_2) \in A$ is chosen with probability $\mu(a)$, then player i is informed of the component a_i .

⁸If this event does not occur, the players communicate infinitely long and then play according to ρ_∞ see below.

Definition 3 An information structure μ on A is a correlated equilibrium of G if and only if

$$\max_{a'_1} \mathbb{E}_{\mu(\mathbf{a}_2|a_1)} g^1(a'_1, \mathbf{a}_2) = \mathbb{E}_{\mu(\mathbf{a}_2|a_1)} g^1(a_1, \mathbf{a}_2)$$

for all $a_1 \in A_1$, and for player 2 similarly.

Theorem 1 For any correlated equilibrium μ of G there exists an extended game $\Gamma_p(T)$ and a π such that P_π is close to μ and π is an ϵ -Nash equilibrium of this extended game.

To state the theorem formally, let us introduce a distance function on ΔA as $d(\mu, \nu) = \max_{a \in A} |\mu(a) - \nu(a)|$.

Then for any ϵ and δ , for any finite $G = (g, A)$ and for any $\mu \in \Delta A$ such that μ is a correlated equilibrium of G , there is $\Gamma_p(T)$ and a π such that $d(P_\pi, \mu) \leq \delta$ and π is an ϵ -Nash equilibrium of the extended game.

4 Proof

We fix a game G and one of its correlated equilibrium distributions μ . We construct a strategy-profile π and then show that it generates a distribution P_π δ -close to μ . Finally we show that π is an ϵ -Nash equilibrium.

We follow the proof through the example introduced in section 2. Let us consider the chicken game again, but now take the following distribution:

μ	0	1
0	$\frac{2}{5}$	$\frac{1}{5}$
1	$\frac{1}{5}$	$\frac{1}{5}$

Fix a G and a $\mu \in \Delta A$ such that μ is a correlated equilibrium of G . Fix $\epsilon > 0$ and $\delta > 0$.

4.1 π

In this section we describe a strategy-profile π . We proceed step by step. First we define σ , then τ and finally ρ .

4.1.1 Lehrer's protocol by the *AND*, the σ

During the mediated phase the players communicate according to σ , which builds up by repetitions of Lehrer's communication strategy. Let us introduce an auxiliary table:

	1	2	3	4
1	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	0(0)
2	$\frac{1}{10}$	$\frac{1}{10}$	0(1)	$\frac{1}{10}$
3	$\frac{1}{10}$	0(2)	$\frac{1}{10}$	0(3)
4	0(4)	$\frac{1}{10}$	0(5)	$\frac{1}{10}$

Number the 0s in the table starting with 0 from the top to the bottom from the left to the right. For example the 0 in the third row fourth column has number 3. There are 6 zeros so let $p = 6$ and define $\sigma_L = (\sigma_0, \dots, \sigma_{p-1})$ as follows. The players randomize over 4 different 6 long message sequences with probability $\frac{1}{4}$. The row player:

R	0	1	2	3	4	5
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	0	1	1	0	0
4	0	0	0	0	1	1

The interpretation of these sequences is that the row player communicates which row she was choosing in the auxiliary table. The players are basically answering yes (1) no (0) questions concerning the 0s in the table. For example if the row player communicates according to 3, she is answering with yes to the 0s numbered 2 and 3. Similarly for the column player:

C	0	1	2	3	4	5
1	0	0	0	0	1	0
2	0	0	1	0	0	0
3	0	1	0	0	0	1
4	1	0	0	1	0	0

By choosing one of the rows and columns in the auxiliary table the players in fact choose an "intended" action. This is given by the following function β for the row player, player 1, and for the column player, player 2:

$$\beta^1(1) = \beta^1(2) = \beta^2(1) = \beta^2(2) = 0, \beta^1(3) = \beta^1(4) = \beta^2(3) = \beta^2(4) = 1.$$

Lehrer's protocol can be describe as follows:

1. Each player chooses a number 1,2,3,4 with probability $\frac{1}{4}$,
2. the players communicate for 6 stages according to the chosen number,
3. if the mediator did not make the public announcement 1, the players play according to β , that is when the communication round of length 6 was successful,
4. if there was a public announcement 1, the players repeat the procedure.

One can see, that the induced distribution by Lehrer's protocol is exactly the desired one. Formally, let $\beta^i : H_p \rightarrow A^i$ interpretations \mathcal{H}_p^i -measurable mappings. We call $\pi_L = (\sigma_L, \beta)$ an L-protocol of length p , where $\sigma_L^i = (\sigma_0^i, \dots, \sigma_l^i, \dots, \sigma_{p-1}^i)$ \mathcal{H}_l^i -measurable mappings, communication strategy under the mechanism *AND*. Clearly there is an induced distribution P_{π_L} by π_L on $(H_p \times A, \mathcal{H}_p \otimes 2^A)$. Denote its marginal on H_p by P_{σ_L} . Denote $S \subset H_p$ the set of plays which contain no $\{*\}$, that is the set of successful rounds. Note that $S \in \mathcal{H}_p^i$ for all i .

Proposition 1 *Lehrer(1991) (JCC): For any finite A and $\mu \in \Delta A$ there is a p and π_L L-protocol, such that $\forall a \in A$:*

$$P_{\pi_L}(\mathbf{a} = a | S) = \mu(\mathbf{a} = a),$$

$$P_{\pi_L}(\mathbf{a}^2 = \cdot | \mathbf{a}^1, S) = P_{\pi_L}(\mathbf{a}^2 = \cdot | \mathcal{H}_p^1 \cap S),$$

The last equality holds if we interchange 1 and 2. Moreover for $i = 1, 2 \exists ! h_p$ such that

$$P_{\sigma_L}(h_p | h_p^i, S^c) = 1.$$

The first condition states that, conditional on the event that the players have not got a $\{*\}$, the induced conditional distribution on A is exactly μ .

This can be seen in the example, if one aggregates the columns and rows according to β .

The second condition states that, on the sub- σ -algebra, where there is no $\{*\}$, \mathbf{a}^1 is a sufficient statistic for \mathbf{a}^2 under P_{π_L} . This means that when a player learns her h_p^i , her information about the other player's action is $\mu(\mathbf{a}^{-i} | \beta^i(h_p^i), S)$, that is exactly the same as she would have learnt her action.

Assume that the row player was communicating according to row 3 and the column player according to column 1 in the auxiliary table. That is the sent messages were:

$$R : 0, 0, 1, 1, 0, 0$$

$$C : 0, 0, 0, 0, 1, 0$$

and the resulting public announcements correspondingly:

$$0, 0, 0, 0, 0, 0.$$

The row player learns that the column player was not choosing the second and the fourth column in the auxiliary table, so she believes that the column player plays with probability 0.5, 0.5 her action 0 or 1. This is exactly $\mu(\mathbf{a}_2|\beta^1(3))$.

The column player learns that the row player was not choosing the fourth row in the auxiliary table, so she believes that the row player plays with probability $\frac{2}{3}$ her action 0 and $\frac{1}{3}$ her action 1. This is exactly $\mu(\mathbf{a}_1|\beta^2(1))$.

Finally the last property of π_L says that whenever a player observed a $\{*\}$, she knows P_{σ_L} almost sure the realized play h_p given her information h_p^i .

Assume that the row player was communicating according to row 4 and the column player according to column 1 in the auxiliary table. That is the sent messages were:

$$R : 0, 0, 0, 0, 1, 1$$

$$C : 0, 0, 0, 0, 1, 0$$

and the resulting public announcements correspondingly:

$$0, 0, 0, 0, 1, 0.$$

That is each player knows which column or row the other has chosen. Thus the players know the sent messages of the other and so they know the resulting play.

As we pointed out in the introductory example, Lehrer's protocol is manipulable. For example player 1 is better off by choosing always 3 or 4 and communicate accordingly and after a successful communication round she can play action 1. However if the players do not play after the first successful round, but they repeat the communication according to σ_L sufficiently many times, these kind of deviations are statistically detectable.

For π , define the communication strategies under the mechanism *AND* by:

$$\sigma = \prod_{k \in \mathbb{N}} \sigma_L,$$

that is, σ builds up of independent repetitions of σ_L .

4.1.2 Direct Talk

After the mediated communication, the players send direct messages simultaneously for $O + \kappa_T p$ stages. In the first O stages the players conduct a jointly controlled lottery which allows them to mark and coordinate on one of the successful rounds.

In the last $\kappa_T p$ stages called the reporting phase, the players reveal their past messages of the mediated communication phase, excluding the round picked by the jointly controlled lottery. This phase is needed to avoid statistically undetectable deviations, such as spying strategies, in the mediated phase.

Let $1 \leq \kappa_1 < \dots < \kappa_T$ denote the first T successful rounds, that is $\kappa_s h \in S$ for $s \in \{1, \dots, T\}$.

Jointly Controlled Lottery

The players want to coordinate on one of the successful rounds and play an action according to their "intended" action β in that round.

Let $O = \log T$ and define the first O coordinates of τ^i as $\tau_t^i(\cdot)(0) = \tau_t^i(\cdot)(1) = 0.5$ for $t \in \{0, \dots, O - 1\}$ whatever the communication history was so far, that is τ^i randomizes uniformly on M^i at first $\log T$ stages independently, no matter what happened in the past. Let $f : H_O = \{\{a\}, \{b\}, \{c\}, \{*\}\}^O \rightarrow \{1, \dots, T\}$ be a surjective mapping such that a and $*$, and b and c are interchangeable.

Proposition 2 *Aumann et. al. (1995)(Jointly controlled lotteries): Such an f function exists and by the first O coordinates of τ induces a uniform distribution on $\{1, \dots, T\}$ in a way that no unilateral deviation from τ affects this distribution.*

Intuitively, the first O stage of the direct talk allows the players to choose jointly and uniformly one of the successful rounds of the mediated communication phase. Denote

$$\kappa^* = \kappa_{f(h_O)}$$

the jointly chosen successful round. If players then play according to

$$\beta(\kappa^* h)$$

their "intended" action in the corresponding successful round, then the protocol induces the same distribution and same information structure as that of Lehrer.

Reporting phase

In this phase of the direct talk the players reveal all their past messages they have sent in the mediated rounds excluding those from the one selected by the jointly controlled lottery. This phase can rule out otherwise non-detectable deviations in the mediated phase.

For example the row player can randomize with 0.25,0.25 communicating according to the first and second row of the auxiliary table and choose a "spying" strategy otherwise. The row player can send message 1 when the question is about the 0(2) and 0(4), if there were no 1 public announcement. Doing so the row player can get the knowledge that the column player is going to play her action 1. Then in another round the row player can send message 1 when the question is about the 0(3) and 0(5), if there were no 1 public announcement. Doing so the row player can get the knowledge that the column player is going to play her action 0.

This kind of deviation cannot be detected statistically because it induces the same distribution μ however the row player information is more than needed for the information structure μ . Notice also that this deviation is not just a shifting in the randomization σ_L^1 but uses communication strategies which are outside of the support σ_L^1 .

If in the reporting phase the players have to reveal their past messages, then a spying player has to lie about her sent messages. These lies can be detected with positive probability. For example when the row player was spying for 0(2) and 0(4), she has to lie but she does not know that the column player was communicating according to column 3 or 4. Then it can be the case, that the column player gets a contradictory report from the row player.

Define $\tau_t^i(\cdot)(m^i) = 1$ if and only if $m_n^i = m^i$ for $t = n + O$ for $\kappa_{Tp} > n \geq 0$ but let $\tau_t^i(\cdot)(0) = \tau_t^i(\cdot)(1) = 0.5$ for $(\kappa^* - 1)p \leq n \leq \kappa^*p - 1$. In words, the players send their past messages of all the mediated communication phase, but they randomize when the corresponding stages are about the successful round chosen by the jointly controlled lottery. We refer to such a direct communication strategy in the reporting phase as *true reporting*.

At the end of the unmediated communication at time $p\kappa_T + O + p\kappa_T$ the players face a play $h = (h_{\kappa_{Tp}}, h_O, h'_{\kappa_{Tp}})$ given their information structures $\mathcal{H}_{\kappa_{Tp}}^i \otimes 2^{H_{0+\kappa_{Tp}}}$. Notice that under a true reporting strategy ${}_k h = {}_k h'$ for $k \neq \kappa^*$. We say that a lie was detected by player i in the reporting phase corresponding to the round in the mediated communication phase different from κ^* if and only if ${}_k h'$ contradicts with ${}_k h^i$. Formally when:

$$P_{\sigma_L}({}_k h' \mid {}_k h^i) = 0$$

Which means that player i has sent ${}_k h^i$ in round k of the mediated communication phase but player $-i$ has sent ${}_k h'^{-i}$ in the reporting phase, when she had to send ${}_k h^{-i}$ what she has sent in fact in the mediated phase. So if ${}_k h'^{-i}$ is incompatible with ${}_k h^i$ according to σ_L then player $-i$ was caught out in a lie.

Let $L^i = \{h \mid \forall k \neq \kappa^*, P_{\sigma_L}({}_k h' \mid {}_k h^i) > 0\}$. That is the set of histories where no lie was detected by player i .

4.1.3 Statistical Testing, $C(\gamma)$

After the reporting phase the players perform a statistical test on the revealed messages of the other. If the empirical frequency of the revealed messages is close to the distribution induced by σ_L the players accept the hypothesis that the other player was communicating according to σ_L .

Now assume that κ_T repetitions of the mediated communication rounds of length p have been made and there were T successful rounds. After the direct communication set $h = (h_{\kappa_T p}, h_O, h'_{\kappa_T p})$ as the realized play.

For an h_p let $I_{h_p}({}_k h') = 1$ if ${}_k h' = h_p$ and 0 otherwise. That is I_{h_p} counts the number of h_p in $h'_{\kappa_T p}$. Set

$$\hat{P}_h(h_p) = \frac{\sum_{k=1, k \neq \kappa^*}^{\kappa_T} I_{h_p}({}_k h')}{\kappa_T - 1}$$

is the empirical relative frequency of h_p in the report $h'_{\kappa_T p}$. This frequency, under the true reporting strategy, corresponds to the distribution $P_{\sigma_L}(h_p)$.

Let us set the players confidence set for some γ as follows.

$$C(\gamma) = \{h \mid \max_{h_p \notin S} |\hat{P}_h(h_p) - P_{\sigma_L}(h_p)| < \gamma\}$$

That is in C player i accepts the hypothesis that, in fact player $-i$ played according to σ^{-i} .

4.1.4 Decision rules, the ρ

Given G let $x^i \in \Delta A^i$ the punishing strategy of player i against player j , that is:

$$x^i = \arg \min_{y^i \in \Delta A^i} \max_{y^j \in \Delta A^j} g^i(y^i, y^j).$$

Define the decision rules as follows for $h = (h_{\kappa_{Tp}}, h_O, h'_{\kappa_{Tp}})$:

$$\rho^i(h) = \beta^i(\kappa_{f(h_O)} h^i).$$

for $h \in L^i \cap C(\gamma)$ and x^i otherwise.

In words, player i plays according to the interpretation β^i in π_L respecting the $s = f(h_O)$ th successful round of the mediated communication (that is the $\kappa^* = \kappa_s$ round) if he accepted the hypothesis that player j was communicating according to σ^j and he was not caught out in a lie in the report phase. Otherwise player i punishes j with x^i .

Proposition 3 π is well defined.

Proof: It is plain by the construction. ■

4.2 Close to μ for the δ

In this section we prove that the induced distribution can be close to μ if the mediated communication phase is long enough. Intuitively, by the law of large numbers the realized history fall in $C(\gamma)$ and so players play according to β . Thus the induced distribution is that of P_{π_L} which by construction equals μ .

Lemma 1 For any γ , there is a $T(\gamma)$ such that $d(P_\pi, \mu) < \delta$

Proof: It is clear that conditional on C the players have expectations $P_\pi(a|C) = \mu(a)$. This follows by Proposition 1 and 2 and by the construction. That is due to the independent repetitions of σ_L and the uniform selection from the successful rounds by τ . Also

$$P_\pi(a) = P_\sigma(C)P_\pi(a|C) + (1 - P_\sigma(C))P_\pi(a|C^c),$$

thus if we can make $P_\sigma(C)$ close to 1, than the P_π will be close to μ .

Notice that κ_T is a negative binomial random variable with success probability $P_{\sigma_L}(S) < 1$ and parameter T . Thus as T increases $\kappa_T - T$ increases as well. So for T big enough by the weak law of large numbers for independent identically distributed random variables (Feller (1971)):

$$P_\sigma(C) > 1 - \delta_3$$

for any δ_3 . In words, both players will accept their hypothesis with arbitrary high $(1 - \delta_3)$ probabilities even if γ is small. Then

$$d(P_\pi, \mu) < \delta$$

for any δ if δ_3 is small enough and T is big enough. ■

4.3 Deviations for the ϵ

In this part of the proof we show that none of the players can gain more than ϵ by deviating from π .

Proposition 4 *Lehrer (1991): Any lie is detected with positive probability.*

Corollary 1 *The players always conduct a true reporting strategy. As a consequence, the only possible deviation in the communication phase is shifting the probabilities of σ_L .*

Now we have to check that:

$$\epsilon \geq \mathbb{E}_{P_{\pi^2, \pi^1}} g^2(a) - \mathbb{E}_{P_\pi} g^2(a).$$

for all π^2 .

We have seen in Proposition 2 and in the corollary above that deviations in the first O stages of τ are useless and that players do not want to lie in the reporting phase. Thus we have to concentrate on deviations in σ and ρ .

4.3.1 Deviation in ρ , given σ

Conditional on a certain play in C , player 1 has expectation:

$$P_{\pi_L}(\mathbf{a}^2 = a^2 | \kappa^* h^1, S) = \mu(\mathbf{a}^2 = a^2 | \beta^1(\kappa^* h)),$$

by the second equality in Proposition 1.

Thus in C players want to follow ρ because μ is a correlated equilibrium. On the other hand $P_\sigma(C^c)$ is very small, so by deviating from the punishing action players cannot gain more than ϵ .

4.3.2 Deviation in σ and ρ

First notice that by the construction, any deviation in σ^2 is equivalent⁹ with independent repetitions of some σ_L^2 . Thus we just deal with deviations in σ_L^2 . Let us write $\sigma' = (\sigma^1, \sigma^2)$.

We need an equivalent definition of a correlated equilibrium:

⁹In expected terms due to player 1 independent repetitions of σ_L^1 and the uniform selection caused by τ !

Lemma 2 $\mu \in \Delta A$ is a correlated equilibrium of G , if and only if the following condition holds for all $r : A^j \rightarrow A^j$ and symmetrically for i :

$$\mathbb{E}_\mu g^j(a) \geq \mathbb{E}_\mu g^j(a^i, r(a^j))$$

We set an appropriate γ for the confidence set and a T to maintain the ϵ -constraint.

Proposition 5 There is a γ and T such that for any deviation $\pi'^2 = (\sigma'^2, \tau'^2, \rho'^2)$:

$$\epsilon \geq \mathbb{E}_{P_{\pi'^2, \pi^1}} g^2(a) - \mathbb{E}_{P_\pi} g^2(a).$$

Proof: For any δ_4 there is a γ and a T such that for any deviation $\pi'^2 = (\sigma'^2, \tau'^2, \rho'^2)$:

$$\max_{\pi'^2} \mathbb{E}_{P_{\pi'^2, \pi^1}} g^2(a) \leq \max_{\nu, r} \mathbb{E}_\nu g^2(a^1, r(a^2))$$

where $r : A^2 \rightarrow A^2$ and $\nu \in \Delta A$ such that $d(\nu, \mu) \leq \delta_4$.

Set $D = \max_{j \in \{1, 2\}} D^j$ and $D^j = \sum_{a \in A} g^j(a)$. Set γ and T such that $\delta_4 < \frac{\epsilon}{2D}$. Then

$$\mathbb{E}_{P_\pi} g^2(a) \geq \mathbb{E}_\mu g^2(a) - \delta_4 D,$$

$$\mathbb{E}_\mu g^2(a^1, r(a^2)) \leq \mathbb{E}_\mu g^2(a)$$

since μ is a CE. Then we have

$$\max_{\nu, r} \mathbb{E}_\nu g^2(a^1, r(a^2)) - \mathbb{E}_\mu g^2(a) \leq \mathbb{E}_{\nu^*} g^2(a^1, r^*(a^2)) - \mathbb{E}_\mu g^2(a^1, r^*(a^2)) \leq \delta_4 D,$$

where ν^*, r^* are the *arg max*. Finally:

$$\frac{\epsilon}{2} + \frac{\epsilon}{2} \geq \mathbb{E}_{P_{\pi'^2, \pi^1}} g^2(a) - \mathbb{E}_{P_\pi} g^2(a).$$

The same is true for player 1. ■ Q.E.D.

5 Conclusion

Given a two-player finite normal form game and one of its correlated equilibrium distribution we defined an extended game, where the players' communication strategies generate the desired distribution as a Nash equilibrium of the extended game.

The extended game have two phases: a mediated one, through the mediator AND , and a

direct communication phase. Each phase has several stages.

Our result is important because with a fix public mediator the players can obtain any correlated equilibria of the original game as Nash equilibria of the extended game.

We hope to consider a similar setup, where the procedure can be interpreted as a bargaining process for correlated equilibria.

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