Optimal Monetary Policy When Agents Are Learning*

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Abstract

Earlier research on optimal monetary policy under learning uses optimality conditions derived under rational expectations. In this paper instead, we derive optimal monetary policy when the central bank knows the algorithm followed by agents to form their expectations and makes active use of the learning behavior. There is a well known intratemporal tradeoff between inflation and output gap stabilization. We show there is also an intertemporal tradeoff generated by the central banks possibility to influence future expectations. The optimal interest rate rule reacts more aggressively to out-of-equilibrium inflation expectations than what would be optimal under rational expectations, as the central bank exploits its possibility to "drive" future expectations closer to equilibrium. Moreover, if beliefs are updated according to recursive least squares, the optimal policy is time-varying.

1 Introduction

A great effort has been recently devoted to the issue of how to design the optimal monetary policy; in particular, the analysis has been concentrated on a dynamic stochastic general equilibrium microfounded framework, where money has real effects due to nominal rigidities[1]. Using this setup, the optimal policy has been derived under rational expectations (RE)[2], and its properties studied[3]. Moreover, the robustness of monetary policy when several strong hypothesis are

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[1]This framework has been called "New Keynesian".

[2]Usually, rational expectations are assumed on both the central bank and the private sector's side.

[3]See Clarida et al. (1999) for a survey on this literature, and Woodford (2003) for an extensive treatise on how to conduct monetary policy *via* interest rate rules.

relaxed is being currently analyzed; in particular, the effects of changing the underlying model[4], or of introducing uncertainty on policymakers' side about key features of the economic environment[5] are active research topics. A potentially important dimension of this robustness analysis concerns how the private sector forms its expectations[6]. In fact, there is an ample empirical evidence[7] suggesting that agents' forecasts are not consistent with the paradigm of full rationality. Moreover, in the last fifteen years[8] the adaptive learning literature has emphasized that imposing RE is not an innocuous assumption, and that the study of the system when the expectations are out of equilibrium is a relevant issue.

There is a growing strand of research on the issues that arise in monetary policy design when agents are not rational, but update their expectations according to some kind of adaptive algorithm[9]. In particular, the main focus has been on the stability under learning of the relevant RE equilibrium, namely, on the possibility to achieve RE as the limit of an adaptive learning scheme[10], when the initial beliefs of the agents are out of equilibrium[11]. Bullard and Mitra (2002) assume that the policy makers follow some Taylor-type rule, and derive the restrictions on the coefficients in the policy rule that yield E-stability. In Evans and Honkapohja (2003a) the central bank conducts monetary policy following the optimality conditions derived when the monetary authority has no commitment device and the private sector has RE; the authors show that stability under learning of the optimal discretionary RE equilibrium is secured when the optimal policy is implemented through an interest rate rule that reacts not only to the fundamental shocks, but also to private sector expectations. In Evans and Honkapohja (2003b) a similar analysis is conducted when the policy makers follow the optimality conditions derived when the monetary authority can credibly commit to the fully optimal (Ramsey) plan, and the private sector has RE. Honkapohja and Mitra (2005) use the same setup developed in Evans and Honkapohja (2003a), but relax the assumption that the private sector and the central bank have homogenous expectations; instead, they study what changes, in terms of E-stability of the system, when the monetary authority implements the desired policy using its internal forecasts, and not the private sector beliefs.

[4]E.g., see Steinsson (2003).

[5]For examples of parameter uncertainty, see Wieland (2000a,b); for data uncertainty, see Aoki (2002) and Orphanides and Williams (2002); for model uncertainty, see Levin et al. (2003) and Hansen and Sargent (2001).

[6]For an early analysis of optimal monetary policy with adaptive expectations, see Phelps (1967). For more recent analysis, see Sargent (1999).

[7]For an early result in this spirit, see Roberts (1997); more recent papers are Forsells and Kenny (2002) and Adam and Padula (2003).

[8]For an early contribution to adaptive learning applied to macroeconomics, see Cagan (1956). For early applications to the Muth market model see Fourgeaud et al. (1986) and Bray and Savin (1986). The modern literature on this topic was initiated by Marcet and Sargent (1989), who were the first to apply stochastic approximation techniques to study the convergence of learning algorithm. For a recent monograph on the adaptive learning paradigm, see Evans and Honkapohja (2001).

[9]See Evand and Honkapohja (2003c) for a recent survey.

[10]The algorithm typically used in this literature is the recursive least squares.

[11]This property is known in the literature as *E-stability*.

Other papers adopt learning algorithms that prevent the agents' beliefs from settling down, but make them oscillate persistently around the relevant RE equilibrium. Examples of this approach are: Sargent (1999), where a misspecification in the central bank model of the economy is coupled with perpetual learning dynamics to rationalize the sharp reduction in the US inflation starting from the Volcker's period; Gaspar and Smets (2002) and Orphanides and Williams (2003), where the focus is on the consequences of nonrational expectations on the optimal degree of conservatism of the central banker.

In this paper, we take a normative approach, and address the issue of how a rational central bank should conduct the monetary policy optimally in a New Keynesian setup, if the private sector is forming its expectations in a way consistent with the adaptive learning literature. Our work is closely related to Evans and Honkapohja (2003a,b); as mentioned above, in these papers the authors show how the policymakers can design an interest rate rule that makes the economy converge asymptotically to the optimal RE equilibrium, if the agents' expectations are nonrational, and that guarantees to achieve the optimal RE equilibrium (and its determinacy), if instead the private sector forms its beliefs according to the RE paradigm. However, in designing this rule, the monetary authority does not take into account how its current decisions affect future expectations of the private sector under learning. Instead, we assume that the central bank knows the algorithm followed by the agents to form their expectations, and take into account its possibility to influence future beliefs. An analogous investigation when the model is characterized by a Phillips Curve à la Lucas is performed in Sargent (1999), Chapter 5.

Both the asymptotic properties and the features along the transition of the resulting policy are studied. A first result is that the private sector's expectations will converge to the rational expectations equilibrium[12]; moreover, if beliefs are updated according to a recursive least squares algorithm, the optimal policy is time-varying, reflecting the fact that the incentives for the central bank to manipulate agents' beliefs evolve over time. Along the transition, the optimal interest rate rule is characterized by a reaction in front of out-of-equilibrium inflation expectations more aggressive than what would be optimal under RE, as a result of the fact that the central bank exploits its possibility to influence future beliefs. We also show that the main results are preserved even if we allow for nonobservability of the fundamental shocks and of the private sector expectations.

The rest of the paper is organized as follows: in Section 2 we analyze the simplest possible model, where there is no exogenous cost-push shock; Section 3 study how the introduction of the cost-push shock affect our results; Section 4 relaxes the assumptions that the policy maker can perfectly observe the fundamental shocks and the beliefs of the agents; Section 5 summarizes and concludes.

[12] In other words, the optimal policy is E-stable.

2 The Model

We will consider the baseline version of the New Keynesian model, which is by now the workhorse in monetary economics; in this framework, the economy is characterized by two structural equations[13]. The first one is an IS equation:

$$x_t = E_t^* x_{t+1} - \sigma^{-1}(r_t - E_t^* \pi_{t+1} - \overline{rr}_t) + g_t \tag{1}$$

where x_t, r_t and π_t denote time t output gap[14], short-term nominal interest rate and inflation, respectively; σ is a parameter of the household's utility function, representing the intertemporal elasticity of substitution, g_t is an exogenous demand shock and $\overline{rr}_t$ is the natural real rate of interest, i.e. the real interest rate that would hold in absence of any nominal rigidity. Note that the operator E_t^* represents the (conditional) agents' expectations, which are not necessarily rational[15]. The above equation is derived loglinearizing the household's Euler equation.

The second equation is the so-called New Keynesian Phillips Curve (NKPC):

$$\pi_t = \beta E_t^* \pi_{t+1} + \kappa x_t \tag{2}$$

where β denotes the subjective discount rate, and κ is a function of structural parameters; this relation is obtained assuming that the supply side of the economy is characterized by a continuum of firms that produce differentiated goods in a monopolistically competitive market, and that prices are staggered à la Calvo: in other words, in each period firm i can reset the price with a constant probability $1-\theta$, and with probability θ it keeps the same price as in the previous period. If firms take this structure into account when deciding the optimal price, it can be shown[16] that the aggregate inflation is given by (2).

The loss function of the Central Bank (CB) is given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \alpha x_t^2) \tag{3}$$

where α is the relative weight put by the CB on the objective of output gap stabilization[17]. Note that, if expectations are rational (i.e., if $E_t^* = E_t$), there is no trade-off between inflation and output gap stabilization; in fact, following Gali

[13]For the details of the derivation of the structural equations of the New Keynesian model see, among others, Yun (1996), Clarida et al. (1999) and Woodford (2003).

[14]Namely, the difference between actual and natural output.

[15]As shown in Preston (2003), the simple substitution of non-rational expectations in reduced-form equations derived from an intertemporal optimizing model with heterogeneous agents solved under rational expectations, is not necessarily an innocuous assumption, since it could determine the violation of the intertemporal budget constraint; however, as argued in Honkapohja et al. (2003), if learning converges, the intertemporal budget constraint is satisfied *ex post*.

[16]See Yun (1996).

[17]As is shown in Rotemberg and Woodford (1997), equation (3) can be seen as a quadratic approximation to the expected household's utility function; in this case, α is a function of structural parameters.

(2003), we can solve forward equation (2) and impose a boundedness condition on π, obtaining:

$$\pi_t = \kappa E_t \sum_{s=0}^{\infty} \beta^s E_t x_{t+s}$$

Therefore, if CB stabilizes output gap in every period, under RE also inflation will be equal to zero every period; moreover, this plan is time-consistent, in the sense that the optimal plan chosen by the CB if optimizing at period $t+1$ will be equal to the continuation of the optimal plan set when optimizing at t. The absence of inflation bias is due to the fact that, differently from Barro and Gordon (1983) and all the subsequent literature, the target for output chosen by the CB is the natural level of output, and not a higher level; in other words, the target for output gap is zero, as shown in (3). To restore an inflation stabilization-output gap stabilization trade-off is necessary to modify the NKPC introducing a so-called cost-push shock[18].

2.1 Constant Gain Learning

We assume that private sector's expectations are formed according to the adaptive learning literature[19]; in particular, we suppose that agents' Perceived Law of Motion (PLM) is consistent with the law of motion that CB would implement under RE: in other words, both inflation and output gap are assumed to be constant, and agents use a learning algorithm to find out this constant. Throughout this subsection we suppose that expectations evolve following a constant gain algorithm:

$$E_t^* \pi_{t+1} \equiv a_t = a_{t-1} + \gamma(\pi_{t-1} - a_{t-1}) \tag{4}$$

$$E_t^* x_{t+1} \equiv b_t = b_{t-1} + \gamma(x_{t-1} - b_{t-1}) \tag{5}$$

where $\gamma \in (0,1)$. To analyze the optimal control problem faced by the CB, we use the standard Ramsey approach, namely we suppose that the policymakers take the structure of the economy (equations (1) and (2)) as given; moreover, we assume that the CB knows how private agents' expectations are formed, and takes into account its possibility to influence the evolution of the beliefs.

Hence, the CB problem can be stated as follows:

$$\min_{\{\pi_t, x_t, r_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \alpha x_t^2)$$
$$\text{s.t. } (1), (2), (4), (5)$$
$$a_0, b_0 \text{ given}$$

To simplify the problem, we substitute the law of motion of a and b in the structural equations, obtaining:

$$x_0 = b_0 - \sigma^{-1}(r_0 - a_0 - \overline{rr}_0) + g_0 \tag{6}$$

$$x_t = b_{t-1} + \gamma(x_{t-1} - b_{t-1}) - \sigma^{-1}(r_t - a_{t-1} - \gamma(\pi_{t-1} - a_{t-1}) - \overline{rr}_t) + g_t, \qquad t \geq 1$$

[18]For a discussion of this point, see Gali (2003).

[19]For an extensive monograph on this paradigm, see Evans and Honkapohja (2001).

and:

$$\begin{aligned} \pi_0 &= \beta a_0 + \kappa x_0 \\ \pi_t &= \beta\left(a_{t-1} + \gamma(\pi_{t-1} - a_{t-1})\right) + \kappa x_t, \qquad t \geq 1 \end{aligned} \tag{7}$$

Thus, the problem for the CB becomes:

$$\min_{\{\pi_t, x_t, r_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \alpha x_t^2) \tag{8}$$
$$\text{s.t. } (6), (7)$$
$$a_0,\, b_0 \text{ given}$$

The necessary conditions for an optimum are, at every $t \geq 0$:

$$\lambda_t^1 = 0 \tag{9}$$

$$2\pi_t - \lambda_t^2 + \beta^2 \gamma \lambda_{t+1}^2 + \frac{\gamma}{\sigma} \beta \lambda_{t+1}^1 = 0 \tag{10}$$

$$2\alpha x_t + \kappa \lambda_t^2 - \lambda_t^1 + \beta \gamma \lambda_{t+1}^1 = 0 \tag{11}$$

(where λ_t^i, $i = 1, 2$ denotes the Lagrange multiplier associated to (6) and (7), respectively), the structural equations (1)-(2) and the laws of motion of private agents' beliefs, (4)-(5). Combining equations (9), (10) and (11), we obtain the following optimality condition:

$$x_t + \frac{\kappa}{\alpha} \pi_t = \beta^2 \gamma x_{t+1} \tag{12}$$

From equation (2) we can see that, if a_t is different from zero, inflation and output gap cannot be set contemporaneously equal to zero as in the RE case; hence, the fact that the expectations are not rational, introduces a trade-off between inflation and output gap stabilization that is not present under RE. In particular, we have the contemporaneous presence of two trade-offs: an intratemporal trade-off between stabilization of inflation at t and output gap at t, determined by the presence of the nonzero term βa_t in the Phillips Curve (2); and an intertemporal trade-off between optimal behavior at t and stabilization of output gap at $t + 1$, which is generated by the possibility for the CB to manipulate future values of a. To isolate the different impacts of these two trade-offs, we can set $\gamma = 0$ in the optimality condition (12), thus obtaining:

$$x_t + \frac{\kappa}{\alpha} \pi_t = 0 \tag{13}$$

which is identical to the optimality condition derived in the RE optimal monetary policy literature when a cost-push shock is introduced in the Phillips Curve, and CB sets the optimal plan taking private sector's expectations as given (i.e., in the discretionary case)[20]. Clarida et al. (1999) describe this relation as implying a 'lean against the wind' policy: in other words, if output gap (inflation)

[20]For example, see Clarida et al. (1999).

is above target, it is optimal to deflate the economy (contract demand below capacity). Let's instead assume that $\gamma > 0$, so that expectations evolve over time, and that the CB takes it into account; then, the optimality condition is again (12).

Hence, for a given positive value of x_t, the optimal disinflation is less harsh with respect to the one implied by (13) -and, consequently, future inflation beliefs are smaller in absolute values- provided that also x_{t+1} is positive[21]. The reason is that, when the CB can manipulate expectations, it renounces to optimally stabilize the economy in period t, in exchange for a reduction in future inflation expectations that allows an ease in the future inflation output gap trade-off embedded in the Phillips Curve.

This concern for future beliefs (which is not present if $\gamma = 0$) can be also seen comparing the optimal allocations when $\gamma > 0$ with those obtained when expectations are constant (or are assumed by the CB to be independent of its policy decisions), in other words when $\gamma = 0$. To derive the former, we can use (2) and (4) to substitute out x_t and a_{t+1} in (12), thus deriving:

$$\pi_t + \frac{\alpha}{\kappa^2}[\pi_t - \beta a_t] - \gamma\beta^2\frac{\alpha}{\kappa^2}[\pi_{t+1} - \beta(a_t + \gamma(\pi_t - a_t))] = 0 \tag{14}$$

Hence, at an optimum, the dynamics of the economy can be summarized stacking equations (4), (5) and (14), obtaining the trivariate system[22]:

$$y_{t+1} = Ay_t \tag{15}$$

where $y_t \equiv [\pi_t, a_t, b_t]'$, and:

$$A \equiv \begin{pmatrix} \frac{\kappa^2+\alpha+\alpha\beta^3\gamma^2}{\alpha\beta^2\gamma} & \frac{-\alpha\beta\left(1-\beta^2\gamma(1-\gamma)\right)}{\alpha\beta^2\gamma} & 0 \\ \gamma & 1-\gamma & 0 \\ \frac{\gamma}{\kappa} & -\frac{\beta\gamma}{\kappa} & 1-\gamma \end{pmatrix}$$

The three boundary conditions of the above system are:

$$\begin{gathered} a_0,\ b_0 \text{ given} \\ \lim_{t\to\infty} |\pi_t| < \infty \end{gathered} \tag{16}$$

The last one is due to the fact that, if there exists a solution to the problem (8) when the possible sequences $\{\pi_t, x_t, r_t\}$ are restricted to be bounded, then this would be the minimizer also in the unrestricted case[23].

Since A is block triangular, its eigenvalues are given by $1-\gamma$ and by the eigenvalues of:

$$A_{11} \equiv \begin{pmatrix} \frac{\kappa^2+\alpha+\alpha\beta^3\gamma^2}{\alpha\beta^2\gamma} & \frac{-\alpha\beta\left(1-\beta^2\gamma(1-\gamma)\right)}{\alpha\beta^2\gamma} \\ \gamma & 1-\gamma \end{pmatrix} \tag{17}$$

[21] In this sense, we can say that the introduction of learning makes the intratemporal trade-off represented by (13) less severe.

[22] Once we have the equilibrium laws of motion for $[\pi_t, a_t, b_t]$, we can use (1) and (2) to derive the equilibrium r_t.

[23] For a proof, see the Appendix.

In the Appendix we show that A_{11} has one eigenvalue inside and one outside the unit circle, which implies (together with $(1-\gamma) \in (0,1)$) that we can invoke the saddle path stability principle to conclude that the system (15)-(16) has one and only one solution. In other words, there exists one and only one $\overline{\pi}_0$ such that the sequence $\{\pi_t\}$ generated by (15) will have the property $\lim_{t\to\infty} |\pi_t| < \infty$. Moreover, note that $y_{1t} \equiv [\pi_t, a_t]'$ does not depend on b_t, so that the subsystem:

$$y_{1t+1} = A_{11} y_{1t}$$

with the initial conditions $y_{10} = [\overline{\pi}_0, a_0]'$ will generate the same sequence $\{\pi_t, a_t\}$ as the system (15); since A_{11} has the saddle path property, we can express the equilibrium law of motion for inflation[24] as:

$$\pi_t = c_\pi^{cg} a_t \tag{18}$$

We provide a characterization of c_π^{cg} in the following Proposition:

Proposition 1 *Let c_π^{cg} be the feedback coefficient defined in (18); then, the following holds:*
-if $\gamma \in (0,1)$, we have that $0 < c_\pi^{cg} < \frac{\alpha\beta}{\alpha+\kappa^2}$;
-if $\gamma = 0$, i.e. if expectations are constant, we have that $c_\pi^{cg} = \frac{\alpha\beta}{\alpha+\kappa^2}$.

Proof. See the Appendix. ■

Using the structural equation (2) we can derive the ALM for the output gap:

$$x_t = c_x^{cg} a_t \tag{19}$$

where:

$$c_x^{cg} = \frac{c_\pi^{cg} - \beta}{\kappa}$$

From Proposition 1 $c_\pi^{cg} < \frac{\alpha\beta}{\alpha+\kappa^2} < \beta$, thus c_x^{cg} in equation (19) is negative. If private sector expects inflation to be positive, the optimal CB response will imply a negative output gap, i.e. the policymaker will contract economic activity (using the interest rate instrument) in order to attain an actual inflation sufficiently smaller than the expected one.

If $\gamma = 0$, it is easy to see that the laws of motion of inflation and output gap are:

$$\pi_t = \frac{\alpha\beta}{\alpha+\kappa^2} a_t$$

and

$$x_t = -\frac{\kappa\beta}{\alpha+\kappa^2} a_t$$

respectively. From Proposition 1, we know that $c_\pi^{cg} < \frac{\alpha\beta}{\alpha+\kappa^2}$ whenever $\gamma > 0$; on the other hand, $c_\pi^{cg} < \frac{\alpha\beta}{\alpha+\kappa^2}$ implies that $c_x^{cg} < -\frac{\kappa\beta}{\alpha+\kappa^2}$. Intuitively, when the CB makes strategic use of agents learning rules, positive inflationary expectations

[24] Following the adaptive learning terminology, we call it the Actual Law of Motion (ALM).

call for an inflation level lower than in the $\gamma = 0$ case, in order to undercut future expectations; to achieve this goal, the CB is ready to pay a short-term cost represented by a wider current output gap.

Combining the IS curve (1) with the ALM for output gap (19), we obtain the interest rate rule that implements the optimal allocation:

$$r_t = \overline{rr}_t + \delta_\pi a_t + \delta_x b_t + \delta_g g_t \tag{20}$$

where:

$$\delta_\pi^{cg} = 1 - \sigma \frac{c_\pi^{cg} - \beta}{\kappa}$$
$$\delta_x^{cg} = \sigma$$
$$\delta_g^{cg} = \sigma$$

Note that the interest rate rule (20) is, in the terminology introduced in Evans and Honkapohja (2003a,b), an *expectations-based reaction function*, which is characterized by a coefficient on inflation expectations bigger than one (since $c_\pi^{cg} < \frac{\alpha\beta}{\alpha+\kappa^2} < \beta$) and decreasing in c_π^{cg}: an optimal ALM for inflation that requires a more aggressive undercutting of inflation expectations (a lower c_π^{cg}) calls for a more aggressive behavior of the CB when it sets the interest rate (a higher coefficient on inflation expectations in the rule (20)). Moreover, the coefficients on b_t and g_t are such that their effects on the output gap in the IS curve are fully neutralized.

As is shown in the Appendix, c_π^{cg} depends on all the structural parameters; in particular, its dependence on the constant gain γ is not necessarily monotonic: in fact, a higher value of γ has two effects on it: on one hand, it increases the effect of current inflation on future expectations, increasing the incentive for the CB to use this influence (i.e., it would determine a lower c_π^{cg}); on the other hand, it reduces the impact of current expectations on future expectations, thus reducing the benefits from a reduction of the expectations, so that there is an incentive to set a higher c_π^{cg}. In Figure 1 we show a numerical example with the calibration found in Woodford (1999), i.e. with $\beta = 0.99$, $\sigma = 0.157$, $\kappa = 0.024$ and $\alpha = 0.04$; in this case, the first effect dominates, so that c_π^{cg} is a monotonically decreasing function of γ.

Asymptotically, the system will converge to the RE equilibrium, with inflation and output gap equal to zero, and so do the corresponding expectations; this can be seen from the autonomous, linear, homogeneous system of first-order difference equations (15). The asymptotic properties of this kind of systems are well-known[25], and with two eigenvalues inside and one outside the unit circle, and the set of boundary conditions (16), we have only one non-explosive solution, which is such that in the long run the system converges to the trivial solution $y_t = 0$.

This rule is moreover safer to use than the optimal rule under RE when the central bank is unsure whether private agents are rational or follow learning. If private agents agents follow rational expectations the above rule still leads to the optimal RE equilibrium, and ensure determinacy of this equilibrium. This can

[25] See for example Agarwal (1992).

be seen substituting the rule (20) into the structural equations (1)-(2), where the expectations are now assumed to be rational; in this case, the economy evolves according to:

$$\begin{pmatrix} x_t \\ \pi_t \end{pmatrix} = \widehat{A} \begin{pmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{pmatrix}$$

where:

$$\widehat{A} = \begin{pmatrix} c_\pi^{cg} - \beta & \kappa^{-1}(c_\pi^{cg} - \beta) \\ \beta\kappa & \beta \end{pmatrix}$$

Since both variables are non-predetermined, the Blanchard and Kahn (1980) technique implies that a necessary and sufficient condition for the determinacy of the equilibrium is that both the eigenvalues of the matrix $\widehat{A}$ are inside the unit circle; this is equivalent to say[26]:

$$\begin{aligned} &|\mu_1 \mu_2| < 1 \\ &|\mu_1 + \mu_2| < |1 + \mu_1 \mu_2| \end{aligned}$$

where μ_1, μ_2 are the eigenvalues of the matrix. In our case, it is easy to see:

$$\begin{aligned} &|\mu_1 \mu_2| = 0 < 1 \\ &|\mu_1 + \mu_2| = c_\pi^{cg} < 1 \end{aligned}$$

so that the determinacy of the RE equilibrium is guaranteed. Moreover, since the equilibrium is $x_t = \pi_t = 0$ in every period, the rule (20) implements the optimal RE equilibrium.

2.1.1 Time Consistency of Optimal Policy

We will now prove that the optimal policy characterized above is time consistent, in the sense of Lucas and Stokey (1983) and Alvarez et al. (2004). The problem (8) solved at t is said to be time consistent for $t+1$ if the continuation from $t+1$ on of the optimal allocation chosen at t solves (8) in $t+1$; moreover, (8) in period zero is time consistent if (8) in period t is time consistent for $t+1$ for all $t \geq 0$. Let's consider the period t version of (6)-(7):

$$\begin{aligned} x_t &= b_t - \sigma^{-1}(r_t - a_t - \overline{rr}_t) + g_t \qquad (21) \\ x_{t+s} &= b_{t+s-1} + \gamma(x_{t+s-1} - b_{t+s-1}) - \sigma^{-1}(r_{t+s} - a_{t+s-1} - \gamma(\pi_{t+s-1} - a_{t+s-1}) - \overline{rr}_{t+s}) + g_{t+s}, \qquad s \geq 1 \end{aligned}$$

and:

$$\begin{aligned} \pi_t &= \beta a_t + \kappa x_t \qquad (22) \\ \pi_{t+s} &= \beta \left(a_{t+s-1} + \gamma(\pi_{t+s-1} - a_{t+s-1})\right) + \kappa x_{t+s}, \qquad s \geq 1 \end{aligned}$$

Period t problem is:

$$\begin{aligned} &\min_{\{\pi_{t+s}, x_{t+s}, r_{t+s}\}_{s=0}^{\infty}} E_t \sum_{s=0}^{\infty} \beta^s (\pi_{t+s}^2 + \alpha x_{t+s}^2) \\ &\text{s.t. } (21), (22) \\ &\qquad a_t,\ b_t \text{ given} \end{aligned}$$

[26] See LaSalle (1986).

At an optimum, it is easy to see that for any $s \geq 0$ the dynamics of the system are described by:

$$y_{t+s+1} = Ay_{t+s}$$

(where A and y are defined as previously) plus the boundary conditions:

$$a_t,\, b_t \text{ given}$$
$$\lim_{s\to\infty} |\pi_{t+s}| < \infty$$

with x and r given by (1) and (2); if we solve the problem at $t+1$, the optimum is characterized by:

$$y_{t+1+s+1} = Ay_{t+1+s}$$

and:

$$a_{t+1},\, b_{t+1} \text{ given}$$
$$\lim_{s\to\infty} |\pi_{t+1+s}| < \infty$$

But, if the initial beliefs a_{t+1} and b_{t+1} are given by the continuation of the period t solution, $a_{t+1}^{(t)}$ and $b_{t+1}^{(t)}$, then the initial condition for inflation that solves the period $t+1$ problem is:

$$\pi_{t+1} = c_\pi^{cg} a_{t+1}^{(t)} = \pi_{t+1}^{(t)}$$

which shows that the continuation of period t solution solves period $t+1$ problem; since t was arbitrary, we conclude that the problem (8) in period zero is time consistent.

2.2 Decreasing Gain Learning

Thus far, we have assumed that agents update their beliefs according to what is known in the literature as a constant gain algorithm, i.e. a rule in which the parameter γ is constant over time; since past data are downweighted, this algorithm is particularly appropriate when agents believe structural changes to occur. If the private sector suppose that the environment is stationary, then it is more reasonable to model their learning behavior with a decreasing gain rule, namely an algorithm of the form:

$$E_t^* \pi_{t+1} \equiv a_t = a_{t-1} + t^{-1}(\pi_{t-1} - a_{t-1}) \tag{23}$$

$$E_t^* x_{t+1} \equiv b_t = b_{t-1} + t^{-1}(x_{t-1} - b_{t-1}) \tag{24}$$

where the only difference with (4)-(5) is the substitution of γ with t^{-1}; an updating scheme of this form is equivalent[27] to estimate inflation and output gap every period with OLS[28]. Under this assumption, the problem of the CB

[27] Under certain conditions on the values used to initialize the algorithm, see Evans and Honkapohja (2001).

[28] Note that, since inflation and output gap are assumed by the learners to be constant, the OLS is just the sample averages of the two.

becomes:

$$\begin{aligned}&\min_{\{\pi_t,x_t,r_t\}_{t=0}^{\infty}} E_0\sum_{t=0}^{\infty}\beta^t(\pi_t^2+\alpha x_t^2)\\ &\text{s.t. } (1),\ (2),\ (23),\ (24)\\ &\quad a_0,\ b_0 \text{ given}\end{aligned}$$

which can be solved in a way analogous to the constant gain case; hence, the dynamics of the system can be summarized by the optimality condition:

$$\pi_t+\frac{\alpha}{\kappa^2}[\pi_t-\beta a_t]-\frac{1}{t+1}\beta^2\frac{\alpha}{\kappa^2}[\pi_{t+1}-\beta(a_t+\frac{1}{t+1}(\pi_t-a_t))]=0 \tag{25}$$

and by the constraints (23)-(24), and must satisfy the boundary conditions (16). Since we have seen that with constant gain the optimal path for inflation can be written in the form $\pi_t=c_\pi^{cg}a_t$, with decreasing gain we will analyze the optimal inflation within the class[29]:

$$\pi_t=c_{\pi t}^{dg}a_t \tag{26}$$

Using the method of undetermined coefficients, we derive that the sequence $\left\{c_{\pi t}^{dg}\right\}$ must satisfy the non-linear, non-autonomous first order difference equation:

$$c_{\pi t}^{dg}=\frac{\alpha\beta\left(1-\beta^2\frac{1}{t+1}\left(1-\frac{1}{t+1}\right)\right)+\alpha\beta^2\frac{1}{t+1}\left(1-\frac{1}{t+1}\right)c_{\pi t+1}^{dg}}{\kappa^2+\alpha+\left(\frac{1}{t+1}\right)^2\alpha\beta^2(\beta-c_{\pi t+1}^{dg})} \tag{27}$$

Of course, there exist infinite sequences that satisfy equation (27), one for each initial value $c_{\pi 0}^{dg}$; however, since the boundary conditions require π_t to stay bounded, we will concentrate on the solutions for (27) that do not explode[30]. It is easy to show the following Proposition:

Proposition 2 *Let* $\left\{c_{\pi t}^{dg}\right\}$ *be defined by (27), and assume it is bounded; then,* $\lim_{t\to\infty} c_{\pi t}^{dg}$ *exists, and is given by:*

$$\lim_{t\to\infty} c_{\pi t}^{dg}=\frac{\alpha\beta}{\alpha+\kappa^2}$$

Moreover, for any $t<\infty$*, we have:*

$$c_{\pi t}^{dg}<\frac{\alpha\beta}{\alpha+\kappa^2}$$

[29] We thank Albert Marcet for calling our attention on the fact that the full optimum does not necessarily fall into this class.

[30] A proof of existence and uniqueness of a bounded solution of (27) is not worked out completely yet.

Proof. See the Appendix. ■

The result stated in Proposition 2 is analogous to the one obtained for the constant gain case (see Proposition 1), and is determined by the same mechanism: as long as the CB can influence future beliefs (that is, as long as $t^{-1} > 0$), it is optimal to react to out-of-equilibrium expectations more aggressively than in the case of constant expectations (when t goes to infinity), in order to undercut future inflation expectations by a larger amount. This relaxes the future inflation output gap trade-off embedded in the Phillips Curve.

The ALM for output gap and nominal interest rate are as follows:

$$x_t = c_{xt}^{dg} a_t \tag{28}$$

$$r_t = \overline{rr}_t + \delta_{\pi t}^{dg} a_t + \delta_x^{dg} b_t + \delta_g^{dg} g_t \tag{29}$$

where:

$$\begin{aligned} c_{xt}^{dg} &= \tfrac{c_{\pi t}^{dg} - \beta}{\kappa} \\ \delta_{\pi t}^{dg} &= 1 - \sigma \tfrac{c_{\pi t}^{dg} - \beta}{\kappa} \\ \delta_x^{dg} &= \sigma \\ \delta_g^{dg} &= \sigma \end{aligned}$$

Note that the coefficient on inflation expectations in the interest rate rule (29) is time-varying, reflecting the fact that the incentives for the CB to manipulate agents' beliefs evolve over time. In Figure 2, we show how this coefficient depends on time when the parameters are calibrated according to Woodford (1999), namely when $\beta = 0.99$, $\sigma = 0.157$, $\kappa = 0.024$ and $\alpha = 0.04$. First of all, note that it is always above its limiting level, as a consequence of the result stated in the second part of Proposition 2. Moreover, it decreases over time. Interestingly, the next Proposition shows that this decreasing behavior of $\delta_{\pi t}^{dg}$ is a robust feature of the model[31].

Proposition 3 *Let* $\delta_{\pi t}^{dg}$ *be* $1 - \sigma \frac{c_{\pi t}^{dg} - \beta}{\kappa}$, *with* $\left\{ c_{\pi t}^{dg} \right\}$ *defined by (27); then, the sequence* $\left\{ \delta_{\pi t}^{dg} \right\}_{t=1}^{\infty}$ *is monotonic decreasing.*

Proof. See the Appendix. ■

To get an intuition, suppose that a structural break occurs[32], agents know that this has happened and try to learn the new equilibrium; in this situation, it is convenient for the CB to react more aggressively to out-of-equilibrium inflation beliefs in the first periods, when its possibilities to influence private expectations are bigger, even at the cost of larger short-term losses in terms of output gap variability. As time passes, the expectations will be influenced to a lesser extent by the last realization of inflation, hence determining a CB reaction that closely resembles the optimizing behavior when policymakers cannot manipulate expectations.

[31] In fact, a similar picture arises using other calibrations widely adopted in the New Keynesian Literature, like those taken from Clarida et al. (2000) and McCallum and Nelson (1999).

[32] An institutional or policy change, an exogenous shock, etc.

The asymptotic behavior of inflation beliefs is given by the following Proposition:

Proposition 4 *Let $\pi_t = c^{dg}_{\pi t} a_t$, where $c^{dg}_{\pi t}$ is given by (27); then, $a_t \to 0$.*

Proof. See the Appendix. ■

Combining this result with the boundedness of $c^{dg}_{\pi t}$, the ALM for inflation and output gap (26) and (28) tell us that both these variables go to zero asymptotically, restoring the RE allocations. Hence, also with decreasing gain learning the optimal RE equilibrium is E-stable when the CB is designing its policy to optimally take into account the effect of its decisions on future beliefs.

3 Introduction of a cost-push shock

In this section we will change the model introducing an additional term in the Phillips Curve, called cost-push shock[33], so that equation (2) becomes:

$$\pi_t = \beta E^*_t \pi_{t+1} + \kappa x_t + u_t \tag{30}$$

where $u_t \sim N(0, \sigma^2_u)$ is a white noise[34]. In the New Keynesian literature, this shock is introduced to generate a trade-off between inflation and output gap stabilization; because of this, π_t and x_t cannot be set contemporaneously equal to zero in every period. Moreover, the full commitment solution of the optimal monetary policy under RE turns out to be time inconsistent, even if the CB does not have a target for output gap larger than zero. Hence, the time-consistent discretionary solution will be suboptimal, giving rise to what is sometimes called as stabilization bias. There is, however, a crucial difference with the traditional inflation bias problem: the discretion and the commitment solution are not only different in the coefficients of the equilibrium laws of motion of aggregate variables, but even the functional form of these laws of motion differs between the two cases; in particular, while under discretion inflation and output gap are linear functions of the cost-push shock only, under commitment an additional dependence on lagged values of output gap is introduced[35].

Thus, when we turn to the issue of designing the optimal monetary policy when agents are learning, an additional problem arises, namely which PLM the agents are learning. For analytical simplicity, in this section we will restrict our attention to the discretionary case. In particular, we assume that agents believe that inflation and output gap are continous invariant functions of the cost-push shock only, $\pi_t = \pi(u_t)$ and $x_t = x(u_t)$[36]; this hypothesis, together with the iid nature of the shock, implies that the conditional and unconditional

[33]For interpretations of this shock, see among others Clarida et al. (1999), Erceg et al. (2000), Woodford (2003).

[34]Note that the cost-push shock is usually assumed to be an AR(1); we instead assume it to be iid to make the problem more easily tractable, see below.

[35]See Woodford (1999), Clarida et al. (1999) and McCallum and Nelson (2000).

[36]In the terminology of Evans and Honkapohja (2001) Chapter 11, the PLM is a noisy steady state.

expectations of inflation and output gap coincides, and are perceived by the agents as constants. Hence, it is natural to assume that agents will estimate them using their sample means or, more generally, using the stochastic recursive algorithms:

$$E_t^*\pi_{t+1} \equiv a_t = a_{t-1} + \gamma_t(\pi_{t-1} - a_{t-1})$$
$$E_t^* x_{t+1} \equiv b_t = b_{t-1} + \gamma_t(x_{t-1} - b_{t-1})$$

which will coincide with (4)-(5) if $\gamma_t = \gamma \in (0,1)$, and with (23)-(24) if $\gamma_t = t^{-1}$.

We can now follow a procedure analogous to the one used in the model without cost-push shock, and derive the optimality conditions:

$$\lambda_t^1 = 0$$
$$2\pi_t - \lambda_t^2 + \beta^2\gamma_{t+1}E_t\lambda_{t+1}^2 + \frac{\gamma_{t+1}}{\sigma}\beta E_t\lambda_{t+1}^1 = 0$$
$$2\alpha x_t + \kappa\lambda_t^2 - \lambda_t^1 + \beta\gamma_{t+1}E_t\lambda_{t+1}^1 = 0$$

which can be summarized in the following expression:

$$x_t + \frac{\kappa}{\alpha}\pi_t = \beta^2\gamma_{t+1}E_t x_{t+1} \tag{31}$$

Combining (31) with the Phillips Curve (30), we obtain:

$$\pi_t + \frac{\alpha}{\kappa^2}[\pi_t - \beta a_t - u_t] - \gamma_{t+1}\beta^2\frac{\alpha}{\kappa^2}E_t[\pi_{t+1} - \beta(a_t + \gamma_{t+1}(\pi_t - a_t)) - u_{t+1}] = 0 \tag{32}$$

If $\gamma_t = \gamma$, the dynamics of the economy can be summarized by the system:

$$\begin{pmatrix} E_t\pi_{t+1} \\ a_{t+1} \\ b_{t+1} \end{pmatrix} = A \begin{pmatrix} \pi_t \\ a_t \\ b_t \end{pmatrix} + \begin{pmatrix} -\frac{\alpha}{\alpha\beta^2\gamma} \\ 0 \\ -\frac{\gamma}{\kappa} \end{pmatrix} u_t \tag{33}$$

(where A is defined as in the previous section), plus the boundary conditions (16). The system (33)-(16) is in the form studied in Blanchard and Kahn (1980), so that we can use their results. In particular, since there are two predetermined variables and one non-predetermined, there exists one and only one solution if and only if A has one eigenvalue outside the unit circle and two inside, which is exactly the condition we already used in the previous section, and which is proved in the Appendix. Moreover, also the system:

$$\begin{pmatrix} E_t\pi_{t+1} \\ a_{t+1} \end{pmatrix} = A_{11} \begin{pmatrix} \pi_t \\ a_t \end{pmatrix} + \begin{pmatrix} -\frac{\alpha}{\alpha\beta^2\gamma} \\ 0 \end{pmatrix} u_t \tag{34}$$

(where A_{11} is defined as in the previous section) respects the Blanchard-Kahn conditions for existence and uniqueness of a (bounded) solution, and this unique solution can be written as[37]:

$$\pi_t = c_\pi^{cg} a_t + d_\pi^{cg} u_t \tag{35}$$

[37] See Blanchard and Kahn (1980), Proposition 1.

Since in the system (33) the first two equations do not depend on b_t, the solution of (34) must also solve (33), where the last row is initialized at b_0, and must respect the boundary conditions; hence, since the (bounded) solution of (33) is unique, it must coincide with (35). Knowing the functional form of the optimal path for inflation, we can now use the method of guess and verify, and obtain:

$$E_t\pi_{t+1} = c_\pi^{cg} a_{t+1}$$

where c_π^{cg} remains to be determined; substituting this expression in the equation (32), and making use of the law of motion of inflation expectations (4), we derive the values of the coefficients c_π^{cg} and d_π^{cg}, which are summarized in the next Proposition.

Proposition 5 *Let the economy evolve according to the system (33)-(16); then the ALM for inflation is:*

$$\pi_t = c_\pi^{cg} a_t + d_\pi^{cg} u_t$$

where c_π^{cg} *is the same given in Proposition 1, and:*

$$d_\pi^{cg} = \frac{\alpha}{\kappa^2 + \alpha + \gamma^2\alpha\beta^2(\beta - c_\pi^{cg})}$$

The ALM for output gap and the interest rate rule are given by:

$$x_t = c_x^{cg} a_t + d_x^{cg} u_t \tag{36}$$

$$r_t = \overline{rr}_t + \delta_\pi^{cg} a_t + \delta_x^{cg} b_t + \delta_g^{cg} g_t + \delta_u^{cg} u_t \tag{37}$$

where:

$$\begin{aligned} c_x^{cg} &= \tfrac{c_\pi^{cg}-\beta}{\kappa} \\ d_x^{cg} &= \tfrac{d_\pi^{cg}-1}{\kappa} \\ \delta_\pi^{cg} &= 1 - \sigma\tfrac{c_\pi^{cg}-\beta}{\kappa} \\ \delta_x^{cg} &= \sigma \\ \delta_g^{cg} &= \sigma \\ \delta_u^{cg} &= -\sigma\tfrac{d_\pi^{cg}-1}{\kappa} \end{aligned}$$

Plugging (35) into (4), we get:

$$\begin{aligned} a_{t+1} &= a_t + \gamma(c_\pi^{cg} - 1)a_t + \gamma d_\pi^{cg} u_t \\ &= \left(1 - \gamma(1 - c_\pi^{cg})\right) a_t + \gamma d_\pi^{cg} u_t \end{aligned}$$

which is a stationary[38] AR(1); thus, as is well-known in the literature on adaptive learning, the contemporaneous presence of random shocks in the ALM and of constant gain specification of the updating algorithm, prevents the expectations from converging asymptotically to a precise value: instead, we have that $a_t \sim N\left(0, \frac{\gamma^2(d_\pi^{cg})^2}{(1-\gamma(1-c_\pi^{cg}))^2}\sigma_u^2\right)$.

[38] In fact, since $0 < c_\pi^{cg} < 1$, it immediately follows that $0 < \left(1 - \gamma(1 - c_\pi^{cg})\right) < 1$.

If $\gamma_t = t^{-1}$, we can again guess that the ALM for inflation is of the form:

$$\pi_t = c_{\pi t}^{dg} a_t + d_{\pi t}^{dg} u_t \tag{38}$$

which implies that $E_t \pi_{t+1} = c_{\pi t+1}^{dg} a_{t+1}$; substituting this expression in the equation (32), and making use of the law of motion of inflation expectations (23), we obtain that the sequences $\left\{c_{\pi t}^{dg}\right\}$ and $\left\{d_{\pi t}^{dg}\right\}$ must satisfy:

$$\begin{aligned} c_{\pi t}^{dg} &= \frac{\alpha\beta\left(1-\beta^2 \frac{1}{t+1}\left(1-\frac{1}{t+1}\right)\right)+\alpha\beta^2 \frac{1}{t+1}\left(1-\frac{1}{t+1}\right)c_{\pi t+1}^{dg}}{\kappa^2+\alpha+\left(\frac{1}{t+1}\right)^2 \alpha\beta^2(\beta-c_{\pi t+1}^{dg})} \\ d_{\pi t}^{dg} &= \frac{\alpha}{\kappa^2+\alpha+\left(\frac{1}{t+1}\right)^2 \alpha\beta^2(\beta-c_{\pi t+1}^{dg})} \end{aligned}$$

Note that the first equation is the same difference equation (27) that defined $\left\{c_{\pi t}^{dg}\right\}$ in the previous section; we again concentrate on the bounded solution of this difference equation. The ALM for output gap and the interest rate rule are given by:

$$x_t = c_{xt}^{dg} a_t + d_{xt}^{dg} u_t \tag{39}$$

$$r_t = \overline{rr}_t + \delta_{\pi t}^{dg} a_t + \delta_x^{dg} b_t + \delta_g^{dg} g_t + \delta_{ut}^{dg} u_t \tag{40}$$

where:

$$\begin{aligned} c_{xt}^{dg} &= \frac{c_{\pi t}^{dg}-\beta}{\kappa} \\ d_{xt}^{dg} &= \frac{d_{\pi t}^{dg}-1}{\kappa} \\ \delta_{\pi t}^{dg} &= 1-\sigma\frac{c_{\pi t}^{dg}-\beta}{\kappa} \\ \delta_x^{dg} &= \sigma \\ \delta_g^{dg} &= \sigma \\ \delta_{ut}^{dg} &= -\sigma\frac{d_{\pi t}^{dg}-1}{\kappa} \end{aligned}$$

Equation (40) shows that the evolution over time of the coefficient on inflation expectations in the reaction function is the same as in the case without cost-push shock, so that its behavior is characterized again by Propositions 2 and 3.

3.1 Comparison with the Discretionary RE Equilibrium

We can now compare the ALM for inflation and output gap in the constant gain case with the RE benchmark; as shown in Clarida et al. (1999), when the cost-push shock is iid the discretionary optimal policy implies that the RE solutions for π_t and x_t are:

$$\pi_t^{RE} = \frac{\alpha}{\kappa^2+\alpha} u_t \tag{41}$$

and:

$$x_t^{RE} = -\frac{\kappa}{\kappa^2+\alpha} u_t \tag{42}$$

respectively. There are several differences between the above equations and (35)-(36) that are worth considering.

First of all, the contemporaneous impact of the shock on the endogenous variable is different between the two cases; in particular, it is easy to show:

$$\begin{array}{l} d_{\pi}^{cg} < \frac{\alpha}{\kappa^2+\alpha} \\ |d_x^{cg}| > \frac{\kappa}{\kappa^2+\alpha} \end{array} \tag{43}$$

In other words, when agents are learning the CB is less willing to react to noisy shocks, in order to make easier for the private sector to learn what is the "true" value of the conditional expectations of inflation, even if it translates into a more volatile output gap. Hence, the impact of a given nonzero shock drives inflation (output gap) closer to (further from) target when agents are learning, relative to the discretionary RE case. Interestingly, this behavior qualitatively resembles the optimal RE equilibrium under commitment within a simple class of policy rules; in fact, as shown in Clarida et al. (1999), if the CB can commit to a policy rule that is a linear funtion of u_t, the solution can be characterized, when compared to the discretionary equilibrium, by inequalities analogous to (43). However, the (constrained) commitment solution differs from the discretionary one only when the cost-push shock is an AR(1); if u -and consequently, the equilibrium processes for inflation and output gap- is iid, the two solutions coincides, since future (rational) expectations of the agents cannot be manipulated by the CB[39].

Moreover, note that (41) and (42) lack any *intrinsic* source of inertia: if the cost-push shock is iid, so are inflation and output gap. Instead, when private sector expectations are backward-looking, a nonzero realization of u_t affects economy in any $T \geq t$, since the inflation beliefs a_T -that enters both (35) and (36)- can be expressed as a function of the sequence of all the past shocks $\{u_\tau\}_{\tau=0}^{T-1}$. This dependence on the past arises with learning because the CB current actions influences future beliefs through (4) and (5) even if the shock is iid.

The coexistence of these two differences in the reaction to a cost-push shock between learning and RE discretionary policy -a milder impact and a longer persistence- is depicted in Figure 3, where we display the impulse response function of inflation to a unit shock under the two regimes. In the optimal RE discretionary policy, inflation rises on impact and immediately reverts to the steady state once the shock dies out. Instead, under learning the policy maker engineers a smaller initial response of inflation; in subsequent periods inflation gradually converges back to the steady state value. Clarida et al. (1999) and Gali (2003) show a similar disinflation path for the Ramsey policy: a smaller initial inflation compared to the discretionary case, in exchange for a more persistent deviation from the steady state later[40][41]. In both instances this

[39] Instead, if expectations are backward-looking, the future beliefs can be manipulated also when the shock is iid, see below.

[40] A difference is that commitment policy under RE engineers a sequence of negative inflation after the first period, while a positive sequence under learning.

[41] This behavior of Ramsey policy leads to welfare gains over discretion due to the convexity of the loss function; this preference for slower but milder adjustment to shocks is at the heart of the stabilization bias.

pattern results from the CB ability to directly manipulate private expectations, even if the channels used are quite different. In fact, under commitment the policy maker uses a *credible promise on the future* to obtain an immediate decline in inflation expectations and thus in inflation; the inertia in the optimal solution is due to the commitments carried from previous periods. Under learning the pattern results from the sluggishness of expectations: the CB influence private sector's belief through its *past actions*, and the inertia comes from the past realizations of the endogenous variables[42]. In this sense, we can say that the possibility to manipulate future private sector expectations through the learning algorithm plays a role similar to a commitment device under RE, hence easing the short-run trade-off between inflation and output gap.

This result resembles the analysis carried out in Sargent (1999), Chapter 5, which shows that in the Phelps problem under adaptive expectations[43], the optimal monetary policy drives the economy close to the Ramsey optimum. Moreover, when the discount factor β equals 1, optimal policy under learning replicates the Ramsey equilibrium. In our case, optimal policy under learning cannot replicate the commitment solution even for β going to 1. This result follows from the particular nature of the gains from commitment; commitment calls for an ALM with a different functional form than the discretionary case[44]. In the Phelps problem, on the other hand, the Phillips Curve is such that the discretion and commitment outcome of inflation has the same functional form, but different coefficients. However, also in our case an increase in the discount factor makes the optimal disinflationary path under learning getting closer to the commitment solution. This can be seen in Table 1, where we summarize the behavior of inflation in response to a unit cost push shock when the model's parameters are calibrated as in Woodford (1999), apart from β which takes several values. As β goes to 1 the initial response of inflation is milder and the path back to the steady state longer.

Table 1: Paths of inflation for different βs after an initial cost push shock

beta	0.6	0.7	0.8	0.9	1.0
1	0.98	0.98	0.98	0.96	0.93
2	0.52	0.60	0.68	0.75	0.77
3	0.33	0.43	0.55	0.66	0.71
10	0.01	0.03	0.09	0.23	0.39
49	0.00	0.00	0.00	0.00	0.02

Woodford (1999) calibration. Cost push shock $u = 1$ in the first period, starting from $a_{-1} = 0$, $\pi_{-1} = 0$, $x_{-1} = 0$, with $\gamma = 0.9$

[42]We observe a smaller initial response of inflation relative to the RE discretionary case because optimal policy reacts less to the cost push-shock to ease private agents learning.

[43]Phelps (1967) formulated a control problem for a natural rate model with rational central bank and private agents endowed with a mechanical forecasting rule, known to the central bank.

[44]See Clarida et al. (1999).

The resemblance of optimal policy under learning to the RE commitment solution yields that the rule derived in this paper can outperform optimal discretionary RE policy even when the agents are rational. We compute welfare losses using Monte Carlo with a cross sectional sample size of 1000, and a simulation length of 10,000 periods. The three most standard calibrations are used: McCallum and Nelson (1999) (McCN), Woodford (1999) (W) and Clarida et al. (2000) (CCG). The calibrated coefficients are as follows: in McCN $\sigma = 6.097$, $\kappa = 0.3$, $\alpha = 1.83$, the Woodford calibration $\sigma = 0.157$, $\kappa = 0.024$, $\alpha = 0.04$ and the CCG calibration $\sigma = 1/4$, $\kappa = 0.075$, $\alpha = .3$ [45]. In all three calibrations $\beta = 0.99$. We define the variable $\widehat{L} \equiv L^{OP}/L^{RE}$, where:

$$L^{OP} \equiv \frac{1}{1000} \sum_{i=0}^{1000} \sum_{t=0}^{10,000} \beta^t \left(\left(\pi_{i,t}^{OP}\right)^2 + \alpha \left(x_{i,t}^{OP}\right)^2 \right)$$

and:

$$L^{RE} \equiv \frac{1}{1000} \sum_{i=0}^{1000} \sum_{t=0}^{10,000} \beta^t \left(\left(\pi_{i,t}^{RE}\right)^2 + \alpha \left(x_{i,t}^{RE}\right)^2 \right)$$

are the ex-post simulated losses, and OP denotes that the variables are simulated using the optimal rule we derived above; the index i refers to the realization of the cost-push shock used. Table 2 shows that for all 3 calibrations when private agents are rational the decreasing gain OP significantly lowers welfare losses compared to the optimal discretionary rule derived under RE. The central bank tries to ease learning of private agents by decreasing the noise in inflation caused by the cost push shock, a behavior similar to the commitment solution under RE within the simple family of policy rules that are restricted to be linear functions of u_t only. Concerning the constant gain rule (Table 3) calibrations show that it gives similar welfare losses as the optimal RE policy for small γs, and bigger welfare losses than the RE rule for bigger γs[46].

Table 2: Ratio of welfare losses using OP and RE under rational expectations

	$\widehat{L} = L^{OP}/L^{RE}$		
	McCN	W	CCG
Rational expectations	0.995	0.996	0.996

McCN: McCallum and Nelson (1999), W: Woodford (1999)
CCG: Clarida et al. (2000)

[45] We adjust the CCG calibration for quarterly data, i.e. both the σ and κ values reported by Clarida et al. (2000) are divided by 4. We would like to thank Seppo Honkapohja for drawing our attention on this difference in calibrations.

[46] Welfare losses due to inflation variation are lower than under RE, yet for high γs OP allows for high variation in the output gap due to the cost push shock.

Table 3: Ratio of welfare losses using OP and RE under rational expectations

$\widehat{L} = L^{OP}/L^{RE}$

Tracking parameter	McCN	W	CCG
0.1	0.9995	0.9995	0.9995
0.2	0.9996	0.9995	0.9995
0.3	1.0001	0.9997	0.9998
0.4	1.0019	1.0004	1.0006
0.5	1.0070	1.0024	1.0031
0.6	1.0200	1.0081	1.0101
0.7	1.0504	1.0240	1.0292
0.8	1.1151	1.0690	1.0803
0.9	1.2387	1.1979	1.2140

McCN: McCallum and Nelson (1999), W: Woodford (1999)
CCG: Clarida et al. (2000)

3.2 Comparison with the EH rule

Evans and Honkapohja (2003a) (EH hereafter) show that it is of upmost importance that the central bank should condition its interest rate rule on private expectations. This guarantees not only determinacy under RE, but also convergence to the RE equilibrium under learning. This section compares their rule to the two rules derived in our paper[47], to show how optimal monetary policy is modified when the CB optimizes taking into account its effect on private expectations. The end of the section shows welfare gains from using the optimal rule.

The EH rule is equal to[48]:

$$r_t = \overline{rr}_t + \delta_\pi^{EH} a_t + \delta_x^{EH} b_t + \delta_g^{EH} g_t + \delta_u^{EH} u_t \tag{44}$$

where:

$$\begin{aligned} \delta_\pi^{EH} &= 1 + \sigma \tfrac{\kappa\beta}{\alpha+\kappa^2} \\ \delta_x^{EH} &= \sigma \\ \delta_g^{EH} &= \sigma \\ \delta_u^{EH} &= \sigma \tfrac{\kappa}{\alpha+\kappa^2} \end{aligned}$$

It is clear that the coefficients on the output gap expectations and on the demand shock are the same in rule (44) as in rule (40), while the other two coefficients are typically different; in particular, we show in the Appendix the following result:

Proposition 6 *Assume that $t < \infty$; then, $\delta_{\pi t}^{dg} > \delta_\pi^{EH}$, and $\delta_{ut}^{dg} > \delta_u^{EH}$. Moreover, we have:*

[47]Throughout this section, we assume that the private sector is learning.

[48]Note that if agents have RE, i.e. if a_t b_t are replaced by $E_t\pi_{t+1}$ and $E_t x_{t+1}$, respectively, the rule (44) implements the optimal RE equilibrium under discretion, and guarantee the determinacy of this equilibrium, see Evans and Honkapohja (2003a).

- $\lim_{t\to\infty} \delta_{\pi t}^{dg} = \delta_{\pi}^{EH}$,
- $\lim_{t\to\infty} \delta_{ut}^{dg} = \delta_{u}^{EH}$.

Intuitively, the more aggressive response of the monetary policy to out-of-equilibrium inflation expectations is due to the fact that when the CB takes into account its possibility to influence agents' beliefs, it optimally chooses to undercut future inflation expectations more than what would do a myopic CB, namely, a monetary authority that is aware only of the intratemporal trade-off between inflation and output gap stabilization at time t[49]. This can be seen also comparing the optimal allocations for inflation implemented by (40) and (44), which are given by (38) and:

$$\pi_t = \frac{\alpha\beta}{\alpha+\kappa^2} a_t + \frac{\alpha}{\kappa^2+\alpha} u_t$$

respectively. From Proposition 2 we know that the feedback coefficient is smaller in the former case than in the latter, in order to undercut inflation expectations more; besides, also the response to the shock is of lesser magnitude when (40) is used instead of (44) (in fact, $c_{\pi t}^{dg} < \frac{\alpha\beta}{\kappa^2+\alpha}$ implies that $d_{\pi t}^{dg} < \frac{\alpha}{\kappa^2+\alpha}$), because the CB is less willing to accommodate noisy shocks, in order to make easier for the private sector to learn what is the long-term value of the conditional expectations of inflation, even at the cost of higher short-term welfare losses.

To get a quantitative feeling of the welfare gains that the use of rule (40) instead of the EH rule implies along the transition to RE, we perform the following experiment: we define the cumulative *ex-post* losses up to time T under the two interest rules as:

$$L_T^{OP} \equiv \sum_{t=0}^{T} \beta^t \left(\left(\pi_t^{OP}\right)^2 + \alpha \left(x_t^{OP}\right)^2 \right)$$

and:

$$L_T^{EH} \equiv \sum_{t=0}^{T} \beta^t \left(\left(\pi_t^{EH}\right)^2 + \alpha \left(x_t^{EH}\right)^2 \right)$$

where the superscripts OP and EH indicates whether the variables are calculated using rule (40) or (44), respectively; then, we compute numerically the value of $\widehat{L}_T \equiv L_T^{OP}/L_T^{EH}$ at different T's. The results for two possible calibrations widely used in the literature are reported in Table 4: where CCG indicates that $\kappa = 0.075$ and $\alpha = 0.3$ as in Clarida et al. (2000), W means that $\kappa = 0.024$ and $\alpha = 0.04$, as in Woodford (1999); $\beta = 0.99$ is common to the two calibrations[50]. The initial values for expectations are $a_0 = 10$ and $b_0 = 1$, and the shocks are drawn from a standardized normal[51].

[49] Note that $\delta_{\pi t}^{dg}$ converges to δ_{π}^{EH}, reflecting the fact that the influence of the CB on future beliefs vanishes asymptotically.

[50] The risk aversion parameter σ does not appear in the reduced form for inflation and output gap, hence it is not calibrated whatsoever.

[51] Using different initial values for agents' beliefs, or a different realization of the cost-push shock process, does not alter our conclusions.

Table 4: Path of welfare loss ratios using OP and EH

	$\widehat{L} = L^{OP}/L^{EH}$	
T	CCG	W
1	1.04	1.03
2	1.00	1.00
3	0.98	0.98
10	0.94	0.95
49	0.92	0.94

The Table shows that in the very first periods rule (40) yields *ex-post* cumulative welfare losses higher than the EH rule; after a short time span, however, our rule starts generating smaller welfare losses[52]. These findings are consistent with our intuition that a CB that follows rule (40) reacts to out-of-equilibrium inflation expectations more aggressively than in the EH case, in order to undercut more future expectations, even if this means allowing a wider output gap in the short run. This implies that in the first periods, when this more aggressive behavior has not generated a pay-off in terms of smaller $|a|$ sufficient to offset the costly output gap variability, our rule performs worse than the EH one; as soon as inflation expectations become small enough, this initial disadvantage is more than compensated. This pattern is magnified by the time-varying behavior of $\delta_{\pi t}^{dg}$ that we characterized above: the coefficient on inflation expectations in (40) is particularly large in the first periods, hence determining large welfare losses in the short run, and large gains from the contraction of $|a|$ in the medium and long run.

The asymptotic properties of the ALM (38)-(39) depend on the limiting behavior of a_t, which is given by the stochastic recursive algorithm:

$$a_{t+1} = a_t + (t+1)^{-1}\left((c_{\pi t}^{dg} - 1)a_t + d_{\pi t}^{dg} u_t\right) \tag{45}$$

We study its properties in the Appendix, where we use the stochastic approximation techniques[53] to prove the following Proposition:

Proposition 7 *Let a_t evolve according to (45); then, $a_t \to 0$ a.s.*

This result, together with the boundedness of $c_{\pi t}^{dg}$, implies that $c_{\pi t}^{dg} a_t$ goes to zero almost surely; moreover, it is easy to see that $d_{\pi t}^{dg} \to \frac{\alpha}{\kappa^2+\alpha}$, so that we can conclude that $\pi_t \to \frac{\alpha}{\kappa^2+\alpha} v$ almost surely, where v is a random variable with the same probability distribution as u_t.

Also the EH reaction function has this E-stability property, as shown in Evans and Honkapohja (2003a); what changes is the speed of convergence to

[52] We report $\widehat{L}_T$ only until period 50; over a longer horizon, the ratio gets smaller.

[53] For an extensive monograph on stochastic approximation, see Benveniste et al.(1990); the first paper to apply these techniques to learning models is Marcet and Sargent (1989).

RE. In particular, Figure 4[54] shows in the top panel that inflation expectations converge faster with our rule than with the EH one. This is a consequence of the result above derived, namely that when the CB does take into account its influence on the learning algorithm, it has an incentive to undercut future inflation beliefs more than in the case in which it doesn't. On the other hand, in the bottom panel of Figure 4 we can see that output gap expectations converge more slowly with our rule than with the EH one. It is due to the presence of the intertemporal tradeoff described in Section 2: to undercut future inflation expectations the CB is ready to pay a short-term cost, represented by a wider current output gap and, consequently, by a slower convergence of b to its RE value.

3.2.1 Welfare analysis

Finally, we perform numerical welfare loss analysis to estimate the potential welfare gain from using the derived optimal rule. Welfare losses are calculated using Monte Carlo with a cross sectional sample size of 1000, and a simulation length of 10,000 periods. We use the McCN, W and CCG calibration mentioned above.

Table 5 reports ratios of simulated welfare losses $\widehat{L}$ defined as in Table 4: the ratio of welfare losses under the assumption that the central bank follows optimal policy under learning L^{OP} and welfare losses when the central bank follows instead the EH rule equation (44), L^{EH}. Table 6 reports the same ratios for constant gain learning.

Table 5: Ratio of welfare losses using OP and EH discretion under decreasing gain learning

	$\widehat{L} = L^{OP}/L^{EH}$		
	McCN	W	CCG
Decreasing gain	0.973	0.990	0.987

McCN: McCallum and Nelson (1999), W: Woodford (1999)
CCG: Clarida et al. (2000)

As awaited, Table 5 and 6 report that OP performs better; significantly lower welfare losses can be attained when the central bank takes into account its effect on private expectations.

The gain in welfare losses is especially high for constant gain learning with high tracking parameters: for $\gamma = 0.9$ the welfare loss of not using the optimal rule is twice as large as under OP. The intuition behind follows from the fact that, in presence of a cost push shock, constant gain learning does not settle down to RE, but converges to a limiting distribution, thus optimal policy should take into account that this limiting variance in expectations causes welfare losses

[54] To obtain Figure 3, we adopted the Woodford (1999) calibration, with the same initial beliefs and the same realization of the cost-push shock process used to produce Table 1.

Table 6: Ratio of welfare losses using OP and EH under constant gain

$\widehat{L} = L^{OP}/L^{EH}$

Tracking parameter	McCN	W	CCG
0.1	0.980	0.989	0.986
0.2	0.933	0.955	0.948
0.3	0.866	0.901	0.889
0.4	0.801	0.840	0.825
0.5	0.722	0.762	0.743
0.6	0.648	0.679	0.659
0.7	0.582	0.591	0.574
0.8	0.526	0.500	0.491
0.9	0.494	0.439	0.439

McCN: McCallum and Nelson (1999), W: Woodford (1999)
CCG:Clarida et al. (2000)

even in the limit[55].

As noted above, when monetary policy conditions on the interaction between expectations and inflation, it has the incentive to drive out-of-equilibrium expectations more aggressively close to zero. This incentive is stronger the higher is γ, since an increase in the tracking parameter (keeping everything else constant) results in a larger variance of inflation expectations and, consequently, in a larger opportunity cost of adopting a suboptimal rule of the form (44)[56]. This is illustrated in Figure 5, which shows that the higher is γ, the higher is the decrease in variance of a under OP compared to EH.

Moreover, it is worth noting that the use of a myopic rule under constant gain learning allows for the autocorrelation of inflation to rise, thus increasing the persistence of a shock's effect on inflation expectations. This problem arises from the relatively weak response to inflation expectations which feeds back to current inflation and, in turn, into subsequent expectations and inflations. The optimal rule's strong feedback to inflation expectations dampens this interaction between inflation and expectations[57]. In our simulations for example, for $\gamma = 0.1$ autocorrelation of inflation was 0.72 with OP and 0.74 for EH, for $\gamma = 0.9$ autocorrelation increased from 0.91 to 0.98.

Table 7 confirms the argument that it is optimal to lower inflation's deviation

[55] It is worth noting that the EH rule is designed to ensure learnability of the optimal RE in a decreasing gain environment, and its performance under constant gain is never considered on the EH paper; however, it can be useful to employ a constant gain version of their rule to illustrate potential advantages of fully optimal monetary policy over a myopic rule.

[56] In fact, it is easy to see that the optimal interest rate rule coefficient on inflation expectations, δ_π^{cg}, is increasing in γ.

[57] It can be easily derived that the autocorrelation of inflation under constant gain with EH is $E\pi_t^{EH}\pi_{t-1}^{EH} = \left(\frac{\alpha\beta}{\alpha+\kappa^2}\right)^2 \left(1-\gamma+\gamma\frac{\alpha\beta}{\alpha+\kappa^2}\right)\sigma^2_{a_{EH}} + \frac{\alpha\beta}{\alpha+\kappa^2}\left(\frac{\alpha}{\alpha+\kappa^2}\right)^2\gamma\sigma_u^2$ while under the optimal rule $E\pi_t^{OP}\pi_{t-1}^{OP} = \left(c_\pi^{cg}\right)^2\left(1-\gamma+\gamma c_\pi^{cg}\right)\sigma^2_{a_{OP}} + c_\pi^{cg}\left(d_\pi^{cg}\right)^2\gamma\sigma_u^2$. We have already seen that $\sigma^2_{a_{OP}} < \sigma^2_{a_{EH}}$, $c_\pi^{cg} < \frac{\alpha\beta}{\alpha+\kappa^2}$ and $d_\pi^{cg} < \frac{\alpha}{\alpha+\kappa^2}$, thus $E\pi_t^{OP}\pi_{t-1}^{OP} < E\pi_t^{EH}\pi_{t-1}^{EH}$.

from the target even at the cost of higher output gap variation. For decreasing gain learning, when the central bank takes into account its influence on private expectations it engineers an inflation variation 1-3 percent lower, even at the cost of of allowing a 1-3 percent higher welfare loss due to output gap variations. Table 8 shows that under constant gain learning this effect is even more pronounced. The higher is the tracking parameter, the higher is the limiting variance of expectations and the more incentive the central bank has to focus on low variance in inflation allowing for an increase in output gap deviation from the flexible price equilibrium. For $\gamma = 0.9$ the central bank engineers a 30 percent lower welfare loss in inflation when it properly conditions on expectation formation, permitting at the same time 4-7 times more variation in output gap.

Table 7: Ratio of welfare losses using OP and EH under decreasing gain learning due to inflation and output gap variations

$\widehat{L} = L^{OP}/L^{EH}$

	McCN	W	CCG
π	0.97	0.99	0.99
x	1.01	1.03	1.03

McCN: McCallum and Nelson (1999), W: Woodford (1999)
CCG: Clarida et al. (2000)

Table 8: Ratio of welfare losses using OP and EH under constant gain learning due to inflation and output gap variations

$\widehat{L} = L^{OP}/L^{EH}$

	Inflation			Output gap		
Tracking parameter	McCN	W	CCG	McCN	W	CCG
0.1	0.97	0.99	0.98	1.09	1.12	1.12
0.2	0.92	0.95	0.94	1.23	1.34	1.32
0.3	0.84	0.89	0.88	1.42	1.63	1.59
0.4	0.76	0.82	0.80	1.68	2.02	1.95
0.5	0.66	0.74	0.71	2.00	2.54	2.42
0.6	0.56	0.64	0.61	2.42	3.30	3.08
0.7	0.47	0.54	0.51	2.94	4.42	4.04
0.8	0.38	0.42	0.40	3.59	6.13	5.43
0.9	0.30	0.31	0.30	4.41	9.10	7.70

McCN: McCallum and Nelson (1999), W: Woodford (1999), CCG: Clarida et al. (2000)

This section has shown that optimal policy under learning is characterized by a more aggressive reaction to out-of-equilibrium expectations and milder reaction to the cost push shock than would be optimal for a myopic CB. For decreasing gain learning it is optimal to react aggressively to out-of-equilibrium expectations in the first periods even at the cost of higher welfare losses, since the policy maker has more possibility to influence expectations than in later periods. Numerical simulations confirmed that optimal policy under learning engineers lower welfare losses compared to myopic policy. Properly conditioning on private agents expectation formation turns out to be especially important in a nonconvergent environment, when agents follow constant gain learning.

4 Extensions

Up to now, we have supposed that the CB perfectly observes all the relevant state variables of the system, namely the exogenous shocks and the agents' beliefs. In this section we show that our main results extend to a more general framework, where either the shocks or the expectations are not observable. In particular, to make the problem non-trivial, throughout this section we modify the structural equations (1) and (30) with the introduction of unobservable shocks, so that the model is now given by:

$$x_t = E_t^* x_{t+1} - \sigma^{-1}(r_t - E_t^* \pi_{t+1} - \overline{rr}_t) + g_t + e_{xt} \tag{46}$$

and:

$$\pi_t = \beta E_t^* \pi_{t+1} + \kappa x_t + u_t + e_{\pi t} \tag{47}$$

where we assume that the CB can observe π_t and x_t only with a lag, and that e_{xt} and $e_{\pi t}$ are independent white noise that are not observable, not even with a lag. The rest of the setup is identical to subsection 3.1.

4.1 Measurement Error in the Shocks

We start with the case in which the monetary authority can observe g_t and u_t only with an error; in particular, we assume that it receives the noisy signals g_t^* and u_t^*, where:

$$\begin{array}{ll} g_t^* = g_t + \epsilon_t, & \epsilon_t \sim N(0, \sigma_\epsilon^2) \\ u_t^* = u_t + \eta_t, & \eta_t \sim N(0, \sigma_\eta^2) \end{array}$$

To make the problem non-trivial, we also assume that the CB can observe π_t and x_t only with a lag. Note that the shocks do not depend on the policy followed by the CB; hence, the *separation principle* applies, namely, the optimization of the welfare criterion and the estimation of the realizations of the shocks can be solved as separate problems. As is well known, the above signal-extraction problem implies that the expected values of the shocks given the signals are[58]:

$$\begin{array}{l} E\left[g_t/g_t^*\right] \equiv E_t^{CB} g_t = \frac{\sigma_g^2}{\sigma_\epsilon^2 + \sigma_g^2} g_t^* \equiv \zeta_g g_t^* \\ E\left[u_t/u_t^*\right] \equiv E_t^{CB} u_t = \frac{\sigma_u^2}{\sigma_\eta^2 + \sigma_u^2} u_t^* \equiv \zeta_u u_t^* \end{array}$$

Moreover, the separation principle implies that certainty equivalence holds in designing the optimal interest rate rule, which turns out to be identical to (40),

[58] E.g., see Hamilton (1994) Chapter ??.

with g_t and u_t replaced by $E_t^{CB} g_t$ and $E_t^{CB} u_t$, respectively:

$$\begin{aligned} r_t &= \overline{rr}_t + \delta_{\pi t}^{dg} a_t + \delta_x^{dg} b_t + \delta_g^{dg} \zeta_g g_t^* + \delta_{ut}^{dg} \zeta_u u_t^* \\ &= \overline{rr}_t + \delta_{\pi t}^{dg} a_t + \delta_x^{dg} b_t + \delta_g^{dg} \zeta_g g_t + \delta_g^{dg} \zeta_g \epsilon_t + \delta_{ut}^{dg} \zeta_u u_t + \delta_{ut}^{dg} \zeta_u \eta_t \end{aligned}$$

We can combine the above equation with (46) and (47) to obtain the ALM for inflation and output gap:

$$\begin{aligned} \pi_t &= \mu_{at}^1 a_t + \mu_g^1 g_t + \mu_\epsilon^1 \epsilon_t + \mu_{ut}^1 u_t + \mu_{\eta t}^1 \eta_t + \kappa e_{xt} + e_{\pi t} \\ x_t &= \mu_{at}^2 a_t + \mu_g^2 g_t + \mu_\epsilon^2 \epsilon_t + \mu_{ut}^2 u_t + \mu_{\eta t}^2 \eta_t + e_{xt} \end{aligned}$$

where:

$$\begin{aligned} &\mu_{at}^1 = c_{\pi t}^{dg}, \quad \mu_{at}^2 = c_{xt}^{dg} \\ &\mu_g^1 = \kappa \left(1 - \zeta_g\right), \quad \mu_g^2 = 1 - \zeta_g \\ &\mu_\epsilon^1 = -\kappa \zeta_g, \quad \mu_\epsilon^2 = -\zeta_g \\ &\mu_{ut}^1 = \left(d_{\pi t}^{dg} - 1\right) \zeta_u + 1, \quad \mu_{ut}^2 = \left(\frac{d_{\pi t}^{dg} - 1}{\kappa}\right) \zeta_u \\ &\mu_{\eta t}^1 = \left(d_{\pi t}^{dg} - 1\right) \zeta_u, \quad \mu_{\eta t}^2 = \left(\frac{d_{\pi t}^{dg} - 1}{\kappa}\right) \zeta_u \end{aligned}$$

As a consequence of the measurement error, inflation and output gap now depend on a wider set of state variables; however, it is easy to see that the main findings of the preceding section go through in this modified environment. First of all, the separation principle trivially implies that when the CB takes into account the effect of its decisions on future beliefs, the optimal policy is more aggressive against out-of-equilibrium inflation expectations, compared to the case in which the private sector's expectations are considered as exogenously given[59]; moreover, the analysis of convergence of learning algorithms to the optimal discretionary RE equilibrium[60] does not change in this modified environment.

4.2 Heterogenous Forecasts

As argued in Honkapohja and Mitra (2005) (HM hereafter), the hypothesis that the CB can perfectly observe private sector's expectations is subject to several criticisms[61]; it is therefore natural to verify the robustness of our results when this assumption is relaxed. In what follows, we assume that the optimal interest rate rule takes the same form as (40), but the agents' forecasts for inflation and output gap, a_t and b_t, are replaced by the CB internal forecasts, a_t^{CB} and b_t^{CB}[62];

[59] For a description of the optimal policy when the CB does not consider its effect on future beliefs, and there is measurement error in the shocks, see Evans and Honkapohja (2003a) section 4.2.

[60] Note that the optimal RE equilibrium is now different from the baseline case, since inflation and output gap depend also on g_t, ϵ_t, η_t, and the unobservable shocks e_{xt} and $e_{\pi t}$.

[61] For example, private expectations and their forecasts produced by different institutions do not necessarily coincide.

[62] This approach is developed in HM, where it is applied to the EH rule and to a simple Taylor rule. Evans and Honkapohja (2003c) use this method in a setup where the CB follows the expectations based interest rule derived in Evans and Honkapohja (2003b).

in particular, we suppose that the CB and the private sector forecasts have the same form, and are updated according to the same algorithm, which is given by (23)-(24). The only difference is given by the initial beliefs. Note that this setup corresponds to a situation where the CB, in solving its optimization problem, knows the adaptive algorithm used by the agents to form their expectations, but cannot observe the actual values of these expectations; instead, the CB has a tight prior on a_0 and b_0[63], and forms its internal forecasts accordingly. Plugging the interest rate rule into the structural equations (46) and (47), we get the ALM:

$$\begin{array}{l} \pi_t = \nu_a^1 a_t + \nu_{a^{CB}t}^1 a_t^{CB} + \nu_b^1 b_t + \nu_{b^{CB}}^1 b_t^{CB} + \nu_{ut}^1 u_t + \kappa e_{xt} + e_{\pi t} \\ x_t = \nu_a^2 a_t + \nu_{a^{CB}t}^2 a_t^{CB} + \nu_b^2 b_t + \nu_{b^{CB}}^2 b_t^{CB} + \nu_{ut}^2 u_t + e_{xt} \end{array} \tag{48}$$

where:

$$\begin{array}{l} \nu_a^1 = \beta + \kappa\sigma^{-1}, \quad \nu_a^2 = \sigma^{-1} \\ \nu_{a^{CB}t}^1 = -\kappa\sigma^{-1}\left(1 - \sigma\frac{c_{\pi t}^{dg} - \beta}{\kappa}\right), \quad \nu_{a^{CB}t}^2 = -\sigma^{-1}\left(1 - \sigma\frac{c_{\pi t}^{dg} - \beta}{\kappa}\right) \\ \nu_b^1 = \kappa, \quad \nu_b^2 = 1 \\ \nu_{b^{CB}}^1 = -\kappa, \quad \nu_{b^{CB}}^2 = -1 \\ \nu_{ut}^1 = d_{\pi t}^{dg}, \quad \nu_{ut}^2 = d_{xt}^{dg} \end{array}$$

Again, our main results are unaffected by this change in the CB information set, both for $t < \infty$ and for $t \to \infty$. In fact, since the parameters in the optimal rule are the same as in rule (40), the results summarized in Proposition 6 are still valid. On the other hand, we can study E-stability of the system extending Proposition 2 in HM to a time-varying environment. In particular, it is easy to show[64]:

Corollary 1 *Consider the model (48); it is E-stable if and only if the corresponding model with homogenous expectations is E-stable.*

Since E-stability of the homogenous expectations model is ensured by Proposition 7, we conclude that also system (48) is E-stable, and it converges to the optimal discretionary RE equilibrium[65].

5 Conclusions

In this paper we analyzed the optimal monetary policy problem faced by a CB that tries to exploit its possibility to influence future beliefs of the agents, when they follow adaptive learning to form their expectations. This issue is potentially relevant since imposing RE is not an innocuous assumption: interest rate rules

[63] In other words, it believes that $a_0 = a_0^{CB}$ and $b_0 = b_0^{CB}$ with probability one, where a_0^{CB} and b_0^{CB} are given.

[64] The proof is available from the authors upon request.

[65] In fact, the system we are analyzing falls into the class for which E-stability and convergence of real time learning are equivalent, see Evans and Honkapohja (2001).

that are optimal under RE may lead to instability under learning. Moreover, if the CB does not take into account that its decisions affect private sector's future expectations through the learning algorithm, even policies designed to ensure convergence of the economy to the optimal RE equilibrium -like those derived in Evans and Honkapohja (2003a,b)- can perform poorly during the (possibly long) transition path.

To begin with, we considered a standard New Keynesian model without shocks in the Phillips Curve, and derived the optimal monetary policy when agents learn according to a recursive algorithm, with the gain either constant or decreasing. In both cases the first best solution, that can be attained under RE, is not feasible anymore, but is restored asymptotically. In fact, the introduction of learning generates two tradeoffs: a standard intratemporal tradeoff between inflation and output gap stabilization, and an intertemporal tradeoff arising from the CB possibility to influence future expectations. The main difference between the constant and decreasing gain specifications is that the policy function is constant in the former case, and time-varying in the latter. As a side result, we showed that the optimal policy is time consistent.

We also studied a model where a cost-push shock is introduced in the Phillips Curve, and derived the expectations-based reaction function that the CB should use to implement the optimal solution. When we compared this reaction function with the corresponding one obtained in Evans and Honkapohja (2003a) under the assumption that the CB does not take into account its possibility to manipulate future beliefs, the result is a more aggressive response of the monetary policy to out-of-equilibrium inflation expectations; this is a consequence of the intertemporal tradeoff mentioned above, which induce the CB to undercut future inflation expectations more than what would be optimal in the EH setup. Asymptotically, the optimal policy derived in the constant gain case never converges to the RE equilibrium, since the stochastic noise in the Phillips Curve has a nonvanishing impact on inflation expectations; on the other hand, in the decreasing gain case the system converges to the optimal (under discretion) RE equilibrium.

A Constant Gain Learning

Lemma 1 *Let the set of all the real bounded sequences be defined as follows:*

$$M^{\infty} \equiv \{\{z_t\} \in R^{\infty} : \{z_t\} \text{ is bounded}\}$$

and let:

$$G \equiv \left\{\{\pi_t, x_t, r_t\} \in M^{\infty} \times M^{\infty} \times M_{+}^{\infty}\right\}$$

If there exists a sequence $\{\pi_t^*, x_t^*, r_t^*\} \in G$ that solves the problem:

$$\min_{\{\pi_t, x_t, r_t\} \in G} E_0 \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \alpha x_t^2)$$
$$\text{s.t. (6), (7)}$$
$$a_0,\ b_0 \text{ given}$$

then $\{\pi_t^*, x_t^*, r_t^*\}$ solves also (8).

Proof. Let $\{\widehat{\pi}_t, \widehat{x}_t, \widehat{r}_t\}$ be an arbitrary unbounded sequence that satisfies the constraints of (8), and such that:

$$\widehat{V} \equiv \sum_{t=0}^{\infty} \beta^t (\widehat{\pi}_t^2 + \alpha \widehat{x}_t^2) < \infty \tag{49}$$

Let $\{\widehat{\pi}_t^n, \widehat{x}_t^n, \widehat{r}_t^n\}$ be defined as:

$$\{\widehat{\pi}_t^n\} \equiv \{\widehat{\pi}_0, \widehat{\pi}_1, ..., \widehat{\pi}_n, \widehat{\pi}_n, \widehat{\pi}_n, ...\}$$

and $\{\widehat{x}_t^n\}$, $\{\widehat{r}_t^n\}$ are defined accordingly to respect the constraints of (8); clearly, $\{\widehat{\pi}_t^n, \widehat{x}_t^n, \widehat{r}_t^n\}$ is bounded, so that:

$$\widehat{V}^n \geq V^*, \qquad \forall n$$

Since this is true for any n, it must be true also in the limit, i.e.:

$$\lim_{n \to \infty} \widehat{V}^n \geq V^*$$

if $\lim_{n \to \infty} \widehat{V}^n$ exists. However, it is easy to see that $\lim_{n \to \infty} \widehat{V}^n = \widehat{V}$; since $\{\widehat{\pi}_t, \widehat{x}_t, \widehat{r}_t\}$ was arbitrary, it proves the statement[66]. ■

Lemma 2 *Let A_{11} be given by equation (17) in the text; then it has an eigenvalue inside and one outside the unit circle.*

[66] Note that the condition (49) can be imposed without any loss of generality, since any $\{\widehat{\pi}_t, \widehat{x}_t, \widehat{r}_t\}$ that does not respect it, for sure cannot do better than $\{\pi_t^*, x_t^*, r_t^*\}$.

Proof. First of all, we recall a result of linear algebra that we will use in the proof, i.e. that a necessary and sufficient condition for a 2 by 2 matrix to have an eigenvalue inside and one outside the unit circle, is that[67]:

$$|\mu_1 + \mu_2| > |1 + \mu_1 \mu_2|$$

where μ_1, μ_2 are the eigenvalues of the matrix; in the case of A_{11}, the above condition can be written equivalently:

$$\frac{\kappa^2 + \alpha + \alpha\beta^3\gamma^2}{\alpha\beta^2\gamma} + 1 - \gamma > 1 + \frac{\kappa^2 + \alpha + \alpha\beta^3\gamma^2}{\alpha\beta^2\gamma}(1-\gamma) + \frac{\alpha\beta\left(1 - \beta^2\gamma(1-\gamma)\right)}{\alpha\beta^2\gamma}\gamma$$

where we have used the fact that the trace is equal to the sum of the eigenvalues, and that the determinant is equal to the product. After simplifying the above inequality, we get:

$$-\gamma > -\gamma\left(\frac{\kappa^2 + \alpha + \alpha\beta^3\gamma^2 - \alpha\beta\left(1 - \beta^2\gamma(1-\gamma)\right)}{\alpha\beta^2\gamma}\right)$$

so that all we have to prove is that:

$$\frac{\kappa^2 + \alpha + \alpha\beta^3\gamma^2 - \alpha\beta\left(1 - \beta^2\gamma(1-\gamma)\right)}{\alpha\beta^2\gamma} > 1$$

Some tedious algebra shows that this is equivalent to the following expression:

$$\kappa^2 + \alpha(1-\beta)\left(1 - \beta^2\gamma\right) > 0$$

which is always true, since β and γ are supposed smaller than one. ■

We now prove Proposition 1. First of all, we can guess that inflation follows the ALM (18)[68] and use the optimality condition (14) and the method of undetermined coefficients to verify that c_π^{cg} must satisfy the following quadratic expression:

$$\gamma^2\alpha\beta^2\left(c_\pi^{cg}\right)^2 - (\kappa^2 + \alpha + \gamma^2\alpha\beta^3 - (1-\gamma)\alpha\beta^2\gamma)c_\pi^{cg} + \alpha\beta - \gamma\alpha\beta^3(1-\gamma) = 0$$

which can be rewritten as follows:

$$c_\pi^{cg} = \frac{\alpha\beta - \gamma\alpha\beta^3(1-\gamma) + \gamma^2\alpha\beta^2\left(c_\pi^{cg}\right)^2}{\kappa^2 + \alpha + \gamma^2\alpha\beta^3 - (1-\gamma)\alpha\beta^2\gamma} \equiv f(c_\pi^{cg})$$

We will prove that the function $f(\cdot)$, defined on the interval $[0,1]$, is a contraction, so that it admits one and only one fixed point; moreover, since the two roots of the quadratic expression have the same sign (it is due to the fact that $\alpha\beta - \gamma\alpha\beta^3(1-\gamma)$ is a positive term), it follows that the other candidate value for c_π^{cg} is greater than one, which is not compatible with the boundary conditions[69].

First of all, we show that $f(\cdot)$, when defined on the interval $[0,1]$, takes values on the same interval.

[67] See LaSalle (1986).

[68] Which we showed in the text that is the functional form that inflation will have at the optimum.

[69] Since it would imply an exploding inflation.

Lemma 3 $f(c_\pi^{cg})$ *is strictly monotone increasing on the interval* $[0,1]$.

Proof. Note that:

$$\frac{\delta f(c_\pi^{cg})}{\delta c_\pi^{cg}} = \frac{2c_\pi^{cg}\gamma^2\alpha\beta^2}{\kappa^2+\alpha+\gamma 2\alpha\beta^3-(1-\gamma)\alpha\beta^2\gamma\delta}$$

which is positive if and only if the denominator is positive:

$$\kappa^2+\alpha+\gamma 2\alpha\beta^3-(1-\gamma)\alpha\beta^2\gamma\delta \lesseqgtr 0$$

After rearranging:

$$\frac{\frac{\kappa^2}{\alpha}+1}{\gamma\beta^2}+\gamma+\gamma\beta \lesseqgtr 1$$

Since $\frac{1}{\gamma\beta^2}>1$ and all other terms are positive the LHS is indeed bigger than 1. Thus we have prooved that $f(c_\pi^{cg})$ is strictly monotone increasing on the interval [0,1]. ■

Lemma 4 $f(c_\pi^{cg}):[0,1]\rightarrow[0,1]$

Proof. Since $f(c_\pi^{cg})$ is strictly monotone increasing it suffices to show that $f(0)>0$ and $f(1)<1$.

$$f(0)=\frac{\alpha\beta-\gamma\alpha\beta^3(1-\gamma)}{\kappa^2+\alpha+\gamma^2\alpha\beta^3-(1-\gamma)\alpha\beta^2\gamma}$$

where the denominator is positive (see the preceding proof), and the numerator is:

$$\alpha\beta-\gamma\alpha\beta^3(1-\gamma)=\alpha\beta(1-\gamma\beta^2(1-\gamma))>0$$

since $\beta,\gamma\in(0,1)$. Thus $f(0)>0$.

$$f(1)\lesseqgtr 1 \qquad \text{if} \qquad (\beta-1)(\frac{1}{\beta}-\gamma\beta)\lesseqgtr\frac{\kappa^2}{\alpha\beta}$$

but $\beta-1<0$ and $\frac{1}{\beta}>1$, $\gamma\beta<1$ thus we have $(\beta-1)(\frac{1}{\beta}-\gamma\beta)<0$. Because the right hand side is positive $f(1)<1$. ■

To show that $f(\cdot)$ is a contraction, it suffices to show that its derivative is bounded above by a number smaller than one: in fact, by the Mean Value Theorem, we now that for any a,b, there exists a c such that:

$$|f(a)-f(b)|\leq|f'(c)|\,|a-b|$$

and if $|f'(x)|\leq M<1$ for any $c\in[0,1]$, we have the definition of a contraction.

Lemma 5 *For any* $x\in[0,1]$, $0<f'(x)\leq f'(1)<1$.

Proof. First of all, note that:

$$f'(x) = \frac{2\gamma^2\alpha\beta^2 x}{\kappa^2 + \alpha + \gamma^2\alpha\beta^3 - (1-\gamma)\alpha\beta^2\gamma}$$

is positive and increasing in x, so that $\max_{x\in[0,1]} f'(x) = f'(1)$; after some algebraic manipulation, we get:

$$f'(1) \lesseqgtr 1 \Longleftrightarrow \beta^2\gamma - 1 + \beta^2\gamma^2(1-\beta) \lesseqgtr \frac{\kappa^2}{\alpha}$$

Since $\beta, \gamma \in (0,1)$, we have:

$$\beta^2\gamma - 1 + \beta^2\gamma^2(1-\beta) < \beta^2\gamma - 1 + 1 - \beta = \beta(\beta\gamma - 1) < 0$$

so that $f'(1)$ will be smaller than one ($\frac{\kappa^2}{\alpha}$ is always positive). ■

Moreover, we prove the following result.

Lemma 6 *Let $f(\cdot)$ be defined as above; then, $f(\frac{\alpha\beta}{\alpha+\kappa^2}) \leq \frac{\alpha\beta}{\alpha+\kappa^2}$.*

Proof. Note that:

$$\begin{aligned} f(\frac{\alpha\beta}{\alpha+\kappa^2}) = & \frac{\alpha\beta - \gamma\alpha\beta^3(1-\gamma)}{\kappa^2 + \alpha + \gamma^2\alpha\beta^3 - (1-\gamma)\alpha\beta^2\gamma} + \\ & + \frac{\gamma^2\alpha\beta^2}{\kappa^2 + \alpha + \gamma^2\alpha\beta^3 - (1-\gamma)\alpha\beta^2\gamma}\left(\frac{\alpha\beta}{\alpha+\kappa^2}\right)^2 \\ & \gtreqless \frac{\alpha\beta}{\alpha+\kappa^2} \end{aligned}$$

if and only if:

$$\frac{\left(\alpha+\kappa^2\right)\left(1-\gamma\beta^2(1-\gamma)\right)}{\kappa^2 + \alpha + \gamma^2\alpha\beta^3 - (1-\gamma)\alpha\beta^2\gamma} + \frac{\gamma^2\alpha\beta^2}{\kappa^2 + \alpha + \gamma^2\alpha\beta^3 - (1-\gamma)\alpha\beta^2\gamma}\frac{\alpha\beta}{\alpha+\kappa^2} \gtreqless 1$$

Since the $\frac{\alpha\beta}{\alpha+\kappa^2} < \beta$, the LHS is smaller than:

$$\frac{\left(\alpha+\kappa^2\right)\left(1-\gamma\beta^2(1-\gamma)\right) + \gamma^2\alpha\beta^3}{\kappa^2 + \alpha + \gamma^2\alpha\beta^3 - (1-\gamma)\alpha\beta^2\gamma}$$

which is (weakly) smaller than one if and only if:

$$\left(\alpha+\kappa^2\right)\left(1-\gamma\beta^2(1-\gamma)\right) + \gamma^2\alpha\beta^3 \leq \kappa^2 + \alpha + \gamma^2\alpha\beta^3 - (1-\gamma)\alpha\beta^2\gamma$$

Simplifying, this condition turns out to be equivalent to:

$$-\kappa^2\gamma\beta^2(1-\gamma) \leq 0$$

which is always true. So, we conclude that:

$$f(\frac{\alpha\beta}{\alpha+\kappa^2}) \leq \frac{\alpha\beta}{\alpha+\kappa^2}$$

■

We are now ready to prove the Proposition.

Proof of Proposition 1. Combining the Lemmas 4 and 5 we obtain that $f(\cdot)$ is a contraction when defined on the interval $[0, 1]$; moreover, by Lemma 6 we get that f, when defined on $[0, \frac{\alpha\beta}{\alpha+\kappa^2}]$, takes values on the same interval. This result, together with Lemma 5 and with the inequality $\frac{\alpha\beta}{\alpha+\kappa^2} < 1$, implies that $f(\cdot)$ is a contraction also when defined on the interval $[0, \frac{\alpha\beta}{\alpha+\kappa^2}]$ and, therefore, that the optimal c_π^{cg} must be between zero and $\frac{\alpha\beta}{\alpha+\kappa^2}$.

Finally, note that when $\gamma = 0$, $f(c_\pi^{cg})$ collapses to $\frac{\alpha\beta}{\kappa^2+\alpha}$, which proves also the last statement of the Proposition. ■

B Decreasing Gain Learning

We start with the proof of Proposition 2.

Proof of Proposition 2. For the first part of the statement, note that:

$$\lim_{t\to\infty} \alpha\beta\left(1 - \beta^2 \frac{1}{t+1}\left(1 - \frac{1}{t+1}\right)\right) = \alpha\beta$$

and that, if $\left|c_{\pi t}^{dg}\right| \leq M < \infty$ for any t, then:

$$\lim_{t\to\infty} \alpha\beta^2 \frac{1}{t+1}\left(1 - \frac{1}{t+1}\right) c_{\pi t+1}^{dg} = \lim_{t\to\infty} \left(\frac{1}{t+1}\right)^2 \alpha\beta^2(\beta - c_{\pi t+1}^{dg}) = 0$$

Putting together these three limits gives $\lim\limits_{t\to\infty} c_{\pi t}^{dg} = \frac{\alpha\beta}{\alpha+\kappa^2}$.

We prove the second part of the statement by contradiction. Assume that there exists a $T < \infty$ such that $c_{\pi T}^{dg} \geq \frac{\alpha\beta}{\alpha+\kappa^2}$; we show that this implies $c_{\pi t}^{dg} \geq \beta$ for any $t > T$. First of all, from Eq. (27) we can write:

$$\frac{\alpha\beta\left(1 - \beta^2 \frac{1}{T+1}\left(1 - \frac{1}{T+1}\right)\right) + \alpha\beta^2 \frac{1}{T+1}\left(1 - \frac{1}{T+1}\right) c_{\pi T+1}^{dg}}{\kappa^2 + \alpha + \left(\frac{1}{T+1}\right)^2 \alpha\beta^2(\beta - c_{\pi T+1}^{dg})} = c_{\pi T}^{dg} \geq \frac{\alpha\beta}{\alpha+\kappa^2}$$

Rearranging and simplifying, this turns out to be equivalent to:

$$\left(1 - \frac{1}{T+1} + \frac{1}{T+1}\frac{\alpha\beta}{\alpha+\kappa^2}\right) c_{\pi T+1}^{dg} \geq \beta\left(1 - \frac{1}{T+1} + \frac{1}{T+1}\frac{\alpha\beta}{\alpha+\kappa^2}\right)$$

so that $c_{\pi T+1}^{dg} \geq \beta$, which implies also that $c_{\pi T+1}^{dg} \geq \frac{\alpha\beta}{\alpha+\kappa^2}$; then, we can apply the above argument to $c_{\pi T+2}^{dg}$ as well and, proceeding by induction, conclude that $c_{\pi t}^{dg} \geq \beta$ for any $t > T$. An immediate consequence is that $\lim\limits_{t\to\infty} c_{\pi t}^{dg} \geq \beta > \frac{\alpha\beta}{\alpha+\kappa^2}$, which is a contradiction with the result stated in first part of the Proposition, namely $\lim\limits_{t\to\infty} c_{\pi t}^{dg} = \frac{\alpha\beta}{\alpha+\kappa^2}$. Hence, we have showed that there is no $t < \infty$ such that $c_{\pi t}^{dg} \geq \frac{\alpha\beta}{\alpha+\kappa^2}$. ■

To prove Proposition 3 we first state and prove the following technical Lemma:

Lemma 7 *Let λ_1 be defined as:*

$$\lambda_1 \equiv \frac{-\omega_1 + \sqrt{\omega_1^2 - 4\omega_2\omega_0}}{2\omega_2}$$

where:

$$\begin{aligned} &\omega_2 \equiv -\beta\gamma^2 \\ &\omega_1 \equiv \tfrac{\kappa^2+\alpha}{\alpha\beta} + \beta^2\gamma^2 - \beta\gamma\left(1-\gamma\right) \\ &\omega_0 \equiv -\big(1 - \beta^2\gamma\left(1-\gamma\right)\big) \end{aligned}$$

when the restrictions on the parameters α, β and κ are the same imposed in the rest of the paper. If $\gamma \in (0, 1/2]$, the following holds:

$$\frac{\partial}{\partial\gamma}\lambda_1 < 0$$

Proof. First of all, note that:

$$\frac{\partial}{\partial\gamma}\left(-\frac{1}{2}\frac{\omega_1}{\omega_2}\right) = \frac{1}{2}\left(\frac{1}{\gamma^2} - 2\frac{\kappa^2+\alpha}{\alpha\beta}\frac{1}{\gamma^3\beta}\right)$$

and:

$$\frac{\partial}{\partial\gamma}\left(\frac{\sqrt{\omega_1^2 - 4\omega_2\omega_0}}{2\omega_2}\right) = -\frac{1}{4}\frac{2\left(\omega_1/\omega_2\right)\frac{\partial}{\partial\gamma}\left(\omega_1/\omega_2\right) - 4\frac{\partial}{\partial\gamma}\left(\omega_0/\omega_2\right)}{\sqrt{\left(\omega_1/\omega_2\right)^2 - 4\omega_0/\omega_2}}$$

where the RHS is equal to:

$$-\frac{1}{4}\frac{2\frac{1}{\gamma}\left(\frac{\kappa^2+\alpha}{\alpha\beta}\frac{1}{\beta\gamma} + \beta\gamma - \left(1-\gamma\right)\right)\left(1 - 2\frac{\kappa^2+\alpha}{\alpha\beta}\frac{1}{\gamma\beta}\right) + \frac{8}{\gamma\beta} - 4\beta}{\sqrt{\left[\frac{\kappa^2+\alpha}{\alpha\beta}\frac{1}{\beta} + \beta\gamma^2 - \gamma\left(1-\gamma\right)\right]^2 - 4\beta^{-1}\gamma^2\left(1 - \beta^2\gamma\left(1-\gamma\right)\right)}}$$

Clearly, a necessary and sufficient condition for $\frac{\partial}{\partial\gamma}\lambda_1$ to be negative is:

$$\frac{\partial}{\partial\gamma}\left(-\frac{1}{2}\frac{\omega_1}{\omega_2}\right) < -\frac{\partial}{\partial\gamma}\left(\frac{\sqrt{\omega_1^2 - 4\omega_2\omega_0}}{2\omega_2}\right)$$

which can be shown (after some tedious algebra) to be equivalent to:

$$\begin{aligned} 0 > &\left\{-\frac{\alpha\beta\left(2-\gamma\beta^2\right)}{2\left(\kappa^2+\alpha\right) - \gamma\alpha\beta^2}\frac{\kappa^2+\alpha}{\alpha\beta}\frac{1}{\beta} + \frac{1}{\beta} + \gamma\left(1-\gamma\right)\left[\frac{\alpha\beta\left(2-\gamma\beta^2\right)}{2\left(\kappa^2+\alpha\right) - \gamma\alpha\beta^2} - \beta\right]\right\} + \\ &+\frac{\alpha\beta\left(2-\gamma\beta^2\right)}{2\left(\kappa^2+\alpha\right) - \gamma\alpha\beta^2}\gamma^2\left\{\frac{\alpha\beta\left(2-\gamma\beta^2\right)}{2\left(\kappa^2+\alpha\right) - \gamma\alpha\beta^2} - \beta\right\} \end{aligned} \quad (50)$$

A sufficient condition for (50) to hold is that each of the two addends in the RHS is negative. We start from the easiest one, which is the last one; note that:

$$\frac{\alpha\beta\left(2-\gamma\beta^2\right)}{2\left(\kappa^2+\alpha\right)-\gamma\alpha\beta^2}-\beta=\frac{2\alpha\beta-\gamma\alpha\beta^3-2\alpha\beta-2\kappa^2\beta+\gamma\alpha\beta^3}{2\left(\kappa^2+\alpha\right)-\gamma\alpha\beta^2}=\frac{-2\kappa^2\beta}{2\left(\kappa^2+\alpha\right)-\gamma\alpha\beta^2}<0$$

For the first one, we have that it has the same sign as:

$$\gamma\beta\kappa^2-\gamma\left(1-\gamma\right)2\beta\kappa^2$$

which is (weakly) negative if and only if:

$$\gamma\leq\frac{1}{2}$$

Reassuming, we have shown that (50) holds -and, consequently, $\frac{\partial}{\partial\gamma}\lambda_1<0$- whenever $\gamma\leq\frac{1}{2}$. ■

An immediate corollary of the above Lemma is the following:

Corollary 2 *Let λ_{1t} be defined as:*

$$\lambda_{1t}\equiv\frac{-\omega_{1t}+\sqrt{\omega_{1t}^2-4\omega_{2t}\omega_{0t}}}{2\omega_{2t}}$$

where:

$$\begin{aligned}\omega_{2t}&\equiv-\beta\left(\tfrac{1}{t+1}\right)^2\\ \omega_{1t}&\equiv\tfrac{\kappa^2+\alpha}{\alpha\beta}+\beta^2\left(\tfrac{1}{t+1}\right)^2-\beta\tfrac{1}{t+1}\left(1-\tfrac{1}{t+1}\right)\\ \omega_{0t}&\equiv-\left(1-\beta^2\tfrac{1}{t+1}\left(1-\tfrac{1}{t+1}\right)\right)\end{aligned}$$

Then, the subsequence $\{\lambda_{1t}\}_{t=1}^{\infty}$ is monotonic increasing.

Proof. First of all, note that λ_{1t} and ω_{it}, $i=1,2,3$, are defined as the correspondent coefficient in the statement of the Lemma, with γ replaced by $(t+1)^{-1}$; hence, $t+1\geq2$ is equivalent to $\gamma\leq1/2$, which implies that λ_{1t} increases as $(t+1)^{-1}$ decreases. ■

We are now ready to prove Proposition 3.

Proof of Proposition 3. First of all, note that $\delta_{\pi t}^{dg}$ is decreasing if and only if $c_{\pi t}^{dg}$ is increasing; hence, we prove the latter statement. Recall that:

$$c_{\pi t}^{dg}=\frac{\alpha\beta\left(1-\beta^2\frac{1}{t+1}\left(1-\frac{1}{t+1}\right)\right)+\alpha\beta^2\frac{1}{t+1}\left(1-\frac{1}{t+1}\right)c_{\pi t+1}^{dg}}{\kappa^2+\alpha+\left(\frac{1}{t+1}\right)^2\alpha\beta^2(\beta-c_{\pi t+1}^{dg})}$$

which means that, for any finite t, we have:

$$c_{\pi t+1}^{dg}=\frac{-\alpha\beta\left(1-\beta^2\frac{1}{t+1}\left(1-\frac{1}{t+1}\right)\right)+\left(\kappa^2+\alpha+\left(\frac{1}{t+1}\right)^2\alpha\beta^3\right)c_{\pi t}^{dg}}{\alpha\beta^2\frac{1}{t+1}\left(1-\frac{1}{t+1}\right)+\alpha\beta^2\left(\frac{1}{t+1}\right)^2c_{\pi t}^{dg}}$$

Since $\alpha\beta^2\frac{1}{t+1}\left(1-\frac{1}{t+1}\right)+\alpha\beta^2\left(\frac{1}{t+1}\right)^2 c_{\pi t}^{dg}$ is a positive expression, $c_{\pi t+1}^{dg}-c_{\pi t}^{dg} \gtreqless 0$ is equivalent to the second order inequality:

$$\omega_{2t}\left(c_{\pi t}^{dg}\right)^2+\omega_{1t}c_{\pi t}^{dg}+\omega_{0t} \gtreqless 0$$

where:

$$\begin{aligned}\omega_{2t} &\equiv -\beta\left(\tfrac{1}{t+1}\right)^2\\ \omega_{1t} &\equiv \tfrac{\kappa^2+\alpha}{\alpha\beta}+\beta^2\left(\tfrac{1}{t+1}\right)^2-\beta\tfrac{1}{t+1}\left(1-\tfrac{1}{t+1}\right)\\ \omega_{0t} &\equiv -\left(1-\beta^2\tfrac{1}{t+1}\left(1-\tfrac{1}{t+1}\right)\right)\end{aligned}$$

Let λ_{1t}, λ_{2t} be the two roots of the above quadratic expression, such that $\lambda_{1t}<\lambda_{2t}$; since ω_{2t},$\omega_{0t}<0$ for any t, and $-(\omega_{1t}/\omega_{2t})$ can be easily shown to be positive, we know that λ_{1t},$\lambda_{2t}>0$ and that:

$$\lambda_{1t}<c_{\pi t}^{dg}<\lambda_{2t} \Longleftrightarrow c_{\pi t+1}^{dg}-c_{\pi t}^{dg}>0$$

It is easy to see that $\lambda_{2t}>\frac{\alpha\beta}{\alpha+\kappa^2}$ for any t, which implies that:

$$\lambda_{1t}<c_{\pi t}^{dg}<\frac{\alpha\beta}{\alpha+\kappa^2} \Longleftrightarrow c_{\pi t+1}^{dg}-c_{\pi t}^{dg}>0$$

since we showed in Lemma 2 that $c_{\pi t}^{dg}<\frac{\alpha\beta}{\alpha+\kappa^2}$ for any finite t. Now assume, for the sake of contradiction, that $c_{\pi t}^{dg}\le\lambda_{1t}$ for some $\tau\ge 1$; then, $c_{\pi\tau+1}^{dg}\le c_{\pi\tau}^{dg}$ and, for Corollary 2, $\lambda_{1\tau+1}>\lambda_{1\tau}$.

Combining these two inequalities yields the conclusion that $c_{\pi\tau+1}^{dg}\le\lambda_{1\tau+1}$. Repeating infinitely many times the preceding line of reasoning implies that a subsequence of $\left\{c_{\pi t}^{dg}\right\}$ moves monotonically away from $\frac{\alpha\beta}{\alpha+\kappa^2}$, so that $\lim_{t\to\infty} c_{\pi t}^{dg}$, if exists, is for sure smaller than $\frac{\alpha\beta}{\alpha+\kappa^2}$, contradicting Lemma 2. This completes the proof. ■

We now prove Proposition 4.

Proof of Proposition 4. Recall that, as shown Lemma 2, we have $\lim_{t\to\infty} c_{\pi t}^{dg}=\frac{\alpha\beta}{\alpha+\kappa^2}$; since $0<\frac{\alpha\beta}{\alpha+\kappa^2}<1$, for any $\overline{C}$ with $\frac{\alpha\beta}{\alpha+\kappa^2}<\overline{C}<1$, there exists a T such that, for any $t\ge T$ we will have $0<c_{\pi t}^{dg}<\overline{C}$; moreover, using the ALM for π_t, the law of motion of inflation expectations after T can be rewritten as[70]:

$$a_{t+1}=a_t+(t+1)^{-1}(c_{\pi t}^{dg}-1)a_t<a_t+(t+1)^{-1}(\overline{C}-1)a_t$$

where the RHS of the inequality converges to zero, as shown in Evans and Honkapohja (2000). It is also easy to show that, $\forall t\ge T$ we have $a_{t+1}\ge 0$; thus, invoking the Policemen Theorem, we conclude that $\lim_{t\to\infty} a_t=0$, i.e. inflation expectations converge to their RE value. ■

[70]Without loss of generality, we are assuming that $a_T>0$; if the opposite were true, a similar argument applies.

Finally, we prove Proposition 7. First of all, we will briefly describe some results of stochastic approximation[71] that we will exploit in the proof.

Let's consider a stochastic recursive algorithm of the form:

$$\theta_t = \theta_{t-1} + \gamma_t Q\left(t, \theta_{t-1}, X_t\right) \tag{51}$$

where X_t is a state vector with an invariant limiting distribution, and γ_t is a sequence of gains; the stochastic approximation literature shows how, provided certain technical conditions are met, the asymptotic behavior of the stochastic difference equation (51) can be analyzed using the associated deterministic ODE:

$$\frac{d\theta}{d\tau} = h\left(\theta(\tau)\right) \tag{52}$$

where:

$$h\left(\theta\right) \equiv \lim_{t\to\infty} EQ\left(t, \theta, X_t\right)$$

E represents the expectations taken over the invariant limiting distribution of X_t, for any fixed θ. In particular, it can be shown that the set of limiting points of (51) is given by the stable resting points of the ODE (52).

Proof of Proposition 7. Note that our equation (45) is a special case of (51), where the technical conditions are easily shown to be satisfied; moreover, it is also easy to see that:

$$h\left(a\right) = \lim_{t\to\infty} (c_{\pi t}^{dg} - 1)a = \left(\frac{\alpha\beta}{\alpha+\kappa^2} - 1\right) a$$

which has a unique possible resting point at $a^* = 0$. Since $\frac{\alpha\beta}{\alpha+\kappa^2} < 1$, we have that a^* is globally stable, which proves the statement. ■

C Comparison with EH Rule

Proof of Proposition 6. First of all, note that:

$$\delta_{\pi t}^{dg} \gtrless \delta_{\pi}^{EH} \Longleftrightarrow \sigma\frac{\beta - c_{\pi t}^{dg}}{\kappa} \gtrless \sigma\frac{\kappa\beta}{\alpha+\kappa^2}$$

where the second inequality can be rewritten as:

$$\frac{\beta}{\kappa} - \frac{\kappa\beta}{\alpha+\kappa^2} \gtrless \frac{c_{\pi t}^{dg}}{\kappa}$$

Rearranging the terms, we get:

$$\delta_{\pi t}^{dg} \gtrless \delta_{\pi}^{EH} \Longleftrightarrow \frac{\alpha\beta}{\alpha+\kappa^2} \gtrless c_{\pi t}^{dg}$$

[71] See Ljung (1977); Benveniste et al. (1990) provide a recent survey.

Since we have shown in Proposition 2 that $t < \infty$ implies $c_{\pi t}^{dg} < \frac{\alpha\beta}{\alpha+\kappa^2}$, we conclude that $\delta_{\pi t}^{dg} > \delta_{\pi}^{EH}$. Using a similar argument, it is easy to show that:

$$\delta_{ut}^{dg} \gtreqless \delta_{u}^{EH} \Longleftrightarrow \frac{\alpha}{\alpha+\kappa^2} \gtreqless d_{\pi t}^{dg}$$

which implies, since $d_{\pi t}^{dg} = \frac{\alpha}{\kappa^2+\alpha+\left(\frac{1}{t+1}\right)^2 \alpha\beta^2(\beta-c_{\pi t}^{dg})} < \frac{\alpha}{\alpha+\kappa^2}$, that $\delta_{ut}^{dg} > \delta_{u}^{EH}$ whenever $t < \infty$. Finally, note that Proposition 2 also showed that $\lim_{t\to\infty} c_{\pi t}^{dg} = \frac{\alpha\beta}{\alpha+\kappa^2}$, which trivially yields $\lim_{t\to\infty} \delta_{\pi t}^{dg} = \delta_{\pi}^{EH}$ and $\lim_{t\to\infty} \delta_{ut}^{dg} \gtreqless \delta_{u}^{EH}$.. ■

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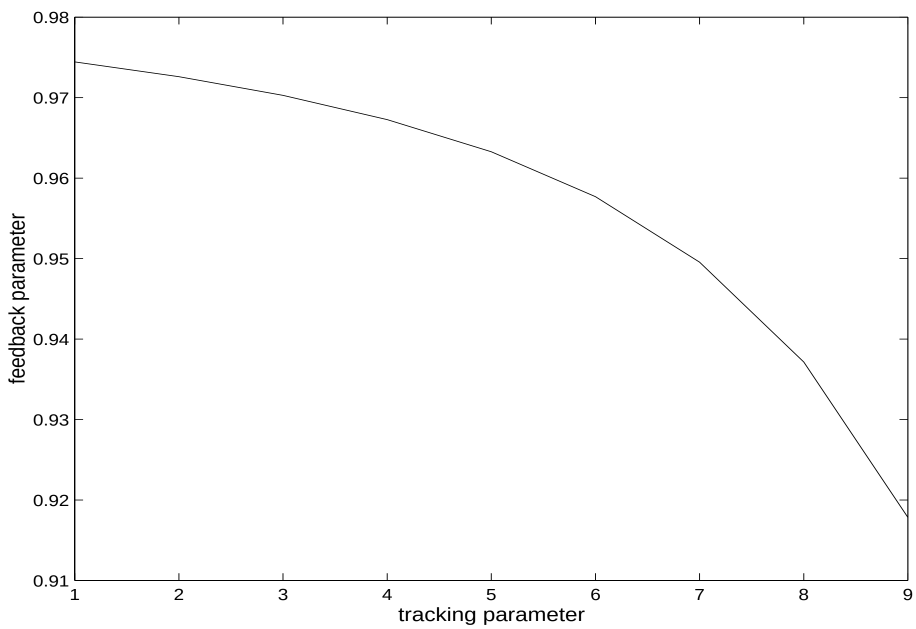

Figure 1: Feedback parameter in the ALM for inflation as a function of γ.

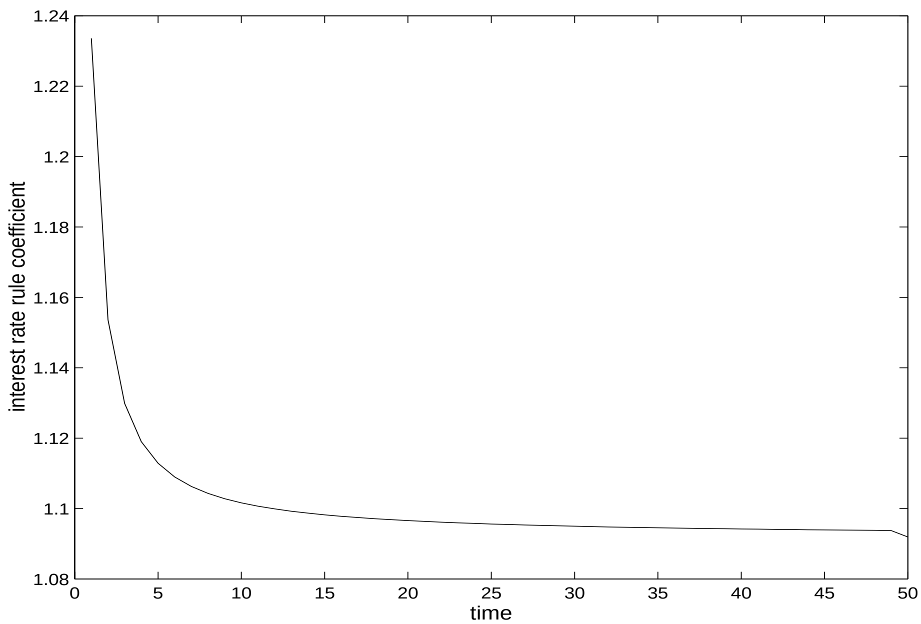

Figure 2: Interest rate rule coefficient on inflation expectations under decreasing gain learning.

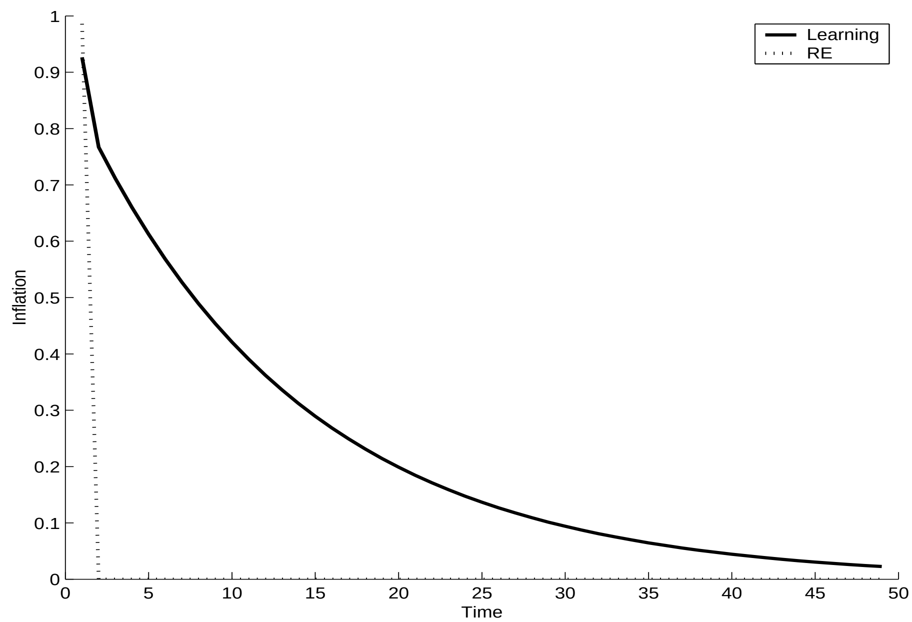

Figure 3: Impulse response of an initial cost-push shock $u = 1$ with optimal policy under learning and optimal discretionary policy under RE, starting from $a_{-1} = 0$, $\pi_{-1} = 0$, $x_{-1} = 0$.

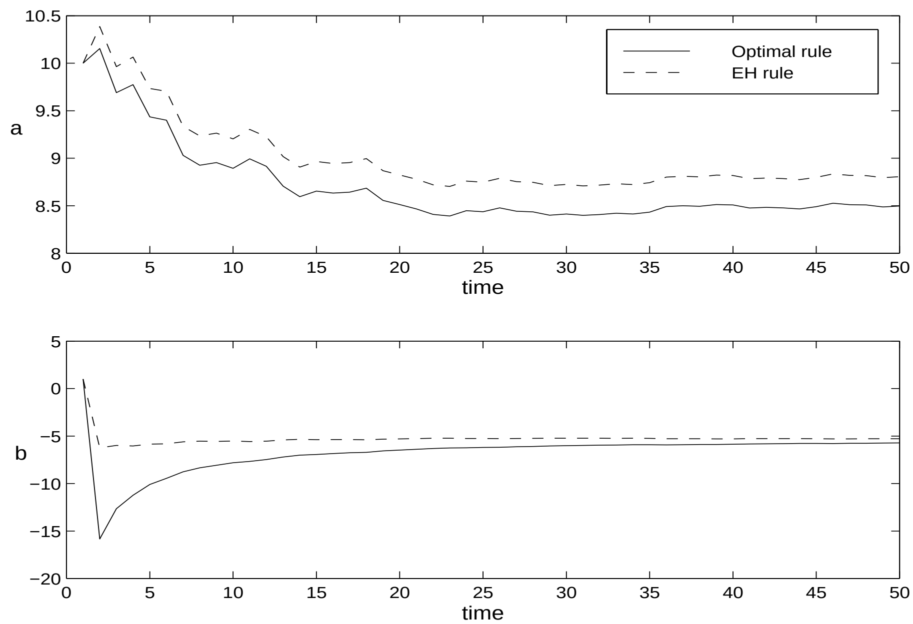

Figure 4: Evolution of inflation and output gap expectations for the optimal and the EH rule, when agents follow decreasing gain learning.

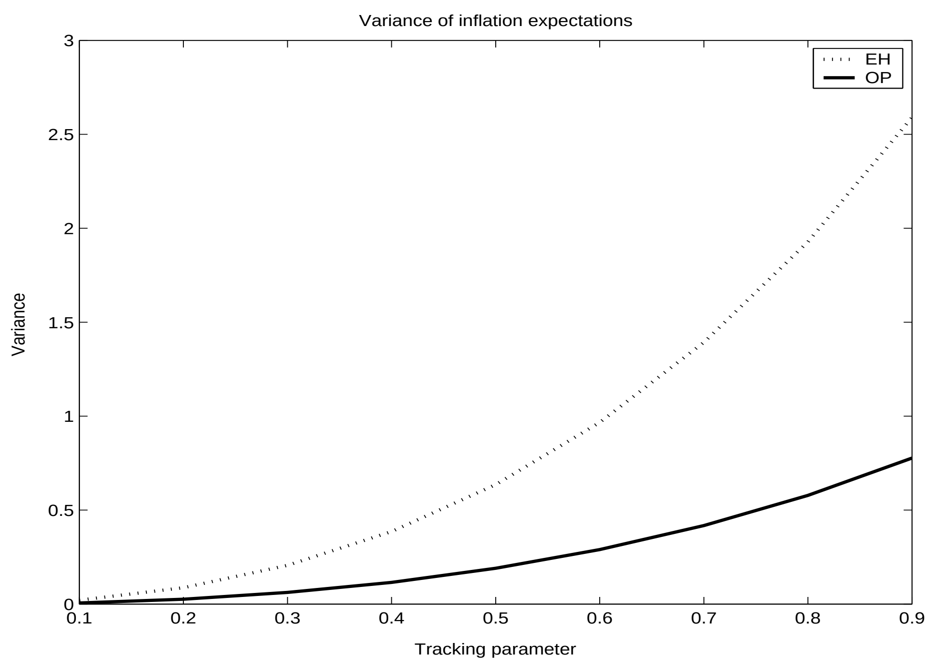

Figure 5: Variance of a_t.