

Endogenous Trading Constraints with Incomplete Asset Markets*

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ABSTRACT. The present paper endogeneizes the borrowing constraints on capital holdings in an infinite horizon incomplete markets model with capital accumulation. In particular, it assumes that two types of households can break their trading arrangements by going into financial autarky, in which case they are seized from any positive asset holdings and excluded from future asset trade forever. Since we show that default cannot be an equilibrium outcome with free entry into a competitive financial intermediation sector, the endogenous trading constraints are then chosen at the loosest possible level that prevents default in equilibrium. Our results show that these limits are significantly different from zero, which is the ad hoc value often introduced in the literature. Further, they are monotone in the aggregate capital stock, since a higher capital reduces the interest rate on the financial obligations and decreases the incentive to default. Finally, in contrast with the existing literature, the limits are non monotone in the idiosyncratic and aggregate shocks due to the presence of incomplete markets and capital accumulation effects.

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1. INTRODUCTION

The present work endogenizes the borrowing constraints by introducing the possibility of default on financial liabilities in an infinite horizon incomplete markets model with capital accumulation. In particular, it studies a heterogeneous agent version of the stochastic growth model where households can decide to break their trading arrangements every period by not paying back their loans. In this case, they will be seized from any positive asset holdings and excluded for future asset trade forever. Further, we show that default in such a framework cannot be an equilibrium outcome in the presence of a competitive financial intermediation sector. Thus, the endogenous trading limits are chosen at the level where households are indifferent between honoring their debt or defaulting at each possible date and state. Using this setup, the main objectives of the present work are to characterize the endogenous trading constraints both analytically and quantitatively. This is done for all levels of the aggregate capital stock along the transitional growth path and in the stationary distribution.

Our work builds a bridge between several important strands of literature. First, it contributes to an active and increasingly growing literature where a number of authors have introduced limited enforceability of risk-sharing contracts, implicitly resulting in agent and state specific trading constraints. In particular, Kehoe and Levine (1993) and Alvarez and Jermann (2000, 2001) introduce these type of limits in exchange economies with a finite number of agents, Krueger and Perri (2001, 2003) study a similar model with a continuum of agents, and Kehoe and Perri (2002, 2004) study a two agent production economy where investors are interpreted as countries. These authors, however, introduce the possibility of default into an otherwise complete markets context, an assumption that has the following critical consequences. First, it allows them to solve a central planning problem without explicitly characterizing the trading limits on assets. Second the incentives for default in these environments are stronger when an agent has a higher idiosyncratic income. This is due to the fact that a positive income shock does not significantly modify the value derived from the risk sharing arrangement when markets are complete, but it does significantly increase the value of autarky. Finally, the models are labelled as endogenous incomplete market economies, since the lack of commitment leads to consumption allocations that do not display perfect risk sharing.

In the present paper, we show that the last result is not robust to the introduction of capital accumulation into closed economies with complete markets similar to the one studied by the previous authors. In particular the equilibrium allocations in the presence of an endogenous production sector exhibit full risk sharing in the long run¹. Since this is clearly at odds with empirical consumption data, it provides a strong motivation for moving towards

¹A similar full risk sharing result is also obtained by Cordoba (2004), who studies a complete markets economy with production and a continuum of types, but with no aggregate uncertainty.

incomplete market economies². In addition, we also show that the positive relationship between income and default incentives may be reversed when markets are incomplete. In this case, the fact that agents are not fully insured against idiosyncratic income shocks implies that a higher income also increases the value of staying in the trading arrangement. Thus, whereas the higher autarky value would lead to higher default incentives, this incomplete markets effect alone would suggest that these incentives are decreasing with income. It turns out that the relative magnitude of the two forces depends on both the aggregate capital stock and the joint distribution of income and wealth. In particular, we find that limits do not vary too much and are non-monotonic in labour income in the stationary distribution. However, along the growth path, the market incompleteness effect dominates hence the limits get typically looser with a higher income

Here, it is also important to note that our incomplete markets assumption implies that we cannot rely on any social planning problem and have to solve for the competitive equilibrium directly. As opposed to the previous complete market economies, however, our decentralized solution provides a complete characterization of the endogenous trading limits (on a limited set of assets) that prevent default in equilibrium. In this respect, our work is also related to the work of Dubey, Geanakoplos and Shubik (2005), where the authors study general equilibrium models with an exogenously incomplete asset structure, ad hoc trading constraints and arbitrary utility penalties upon default. In contrast to this, we chose a particular and economically relevant default punishment³. Further, we endogeneize the borrowing constraints and the value of the outside option, since current and future labour earnings depend on the aggregate capital stock. On the other hand, we do not allow for default in equilibrium. In section (3) we show that if competitive financial intermediaries can choose equilibrium borrowing limits, they will choose the ones which prevent default. The key difference between our approach and the one adopted by the previous work is that they give such market power to the intermediaries that they can charge a different interest rate on loans and deposits. This assumption is critical to support default in equilibrium.⁴

²Note that Kehoe and Perri (2002, 2004) study a model with complete markets and capital accumulation, obtaining an imperfect risk sharing allocation. The “idiosyncratic” shocks are interpreted and calibrated as country specific *aggregate* productivity shocks in their model, however, they are shocks to *individual* labour productivity in our economy. In addition, in their model, the autarky allocation is equivalent to the the optimal allocation of the standard stochastic growth model, that is they can use the capital stock to smooth out aggregate fluctuations, while in our model they can only rely on their individual labour income in autarky.

³Most U.S. household bankruptcies are filed under Chapter 7. Under this procedure, they are seized from any positive asset holdings but can keep their labour income. Whereas they are allowed to borrow after some periods, this becomes considerably more difficult and costly because their credit rating deteriorates significantly.

⁴The welfare implications of a related model with incomplete markets and equilibrium default is studied by Mateos-Planas and Seccia (2005).

Finally, our work is also closely related to the traditional incomplete market models where occasionally binding short-selling or borrowing limits on the different assets are ad hoc. Among others, Heaton and Lucas (1996), Marcet and Singleton (2001) and Telmer (1993) study two agent endowment economies, and Aiyagari (1994), Huggett (1997) and Krusell and Smith (1997, 1998) study production economies with a large number of households. Whereas the previous authors have often argued that the ad hoc trading constraints are tighter than the natural borrowing limits to avoid default in equilibrium, the present work is to our knowledge the first one, with the exception of the papers by Zhang that we discuss in what follows, formalizing this argument. Given this, our work provides a deeper foundation of the trading limits. As already mentioned, the present work is also closely related to the papers of Zhang (1997a, 1997b), where the author derives an endogenous but fixed borrowing limit resulting from the possibility of default in a Lucas type exchange economy with trade in one asset. In contrast to this, our framework characterizes the dependence of the limits on key individual (idiosyncratic income) and aggregate (capital and productivity) variables. Further, it allows for the possibility of capital accumulation, an assumption that considerably modifies the incentives to default under complete and incomplete markets.

As to the characterization of the trading constraints, we show that, in a model where two types of households can trade in physical capital, the endogenous limits that prevent default are significantly different from zero, which is the ad hoc value often assumed in the literature. Moreover, their magnitude in the stationary distribution is reasonable in empirical terms, since it implies that households can borrow between 22 and 63 percent of their annual labour earnings. Further, we find that the limits increase monotonically with a higher aggregate capital stock, since this reduces the interest rate on the financial obligations and it therefore decreases the incentives to default. Given this, we should expect to observe looser borrowing limits and more consumption smoothing as economies grow. Therefore, this mechanism may provide an alternative explanation as to why the increase in earnings inequality has not been accompanied by an increase in consumption inequality in recent decades, as documented by Krueger and Perri (2003).

Finally, we would also like to point out that the presence of endogenous trading limits considerably complicates our computations, since we have to extend usual policy function iteration algorithm to incorporate a state dependent and non rectangular grid for some of the endogenous states, introducing an additional fixed point problem. In addition, since the limits are endogenously determined at the level where the value function from staying in the contract is at least as large as the autarky value, we needed to obtain a good approximation of the value functions close to the borrowing limits. In spite of the computational difficulties, however, we believe the methods developed in the present work can be fruitfully applied to study a wide set of interesting incomplete market models with endogenous limits. In

particular, our results suggests that fiscal policy and social insurance programs can have significant effects on the level of the endogenous trading constraints. Given this, a welfare analysis of any policy reform should take this into account, an important issue that we leave for further research.

The rest of the paper is organized as follows. The following section presents model with incomplete markets, and section three discusses a financial intermediation structure that supports the endogenous limits preventing default in equilibrium. Further, the complete markets allocation and the solution methodology are discussed in sections four and five, and section six presents the quantitative results. Section seven concludes.

2. THE MODEL

We consider an infinite horizon economy with endogenous production, aggregate uncertainty, idiosyncratic labor income shocks and sequential asset trade. Time is discrete and indexed by $t = 0, 1, 2, \dots$. Further, the resolution of uncertainty is represented by an information structure or event-tree S . Each node or date-state $s^t \in S$, summarizing the history of the environment through and including date t , has a finite number of immediate successors, denoted by $s^{t+1}|s^t$. We use the notation $s^r|s^t$ with $r \geq t$ to indicate that node s^r belongs to the sub-tree with root s^t . Further, with the exception of the unique root node s^0 at date $t = 0$, each node has a unique predecessor, denoted by s^{t-1} . The probability as of period 0 of date-event s^t is denoted by $\pi(s^t)$, with $\pi(s^0) = 1$, since the initial realization s^0 is given. In addition, $\pi(s^r|s^t)$ denotes the probability of s^r given s^t , with $\pi(s^t|s^t) = 1$.

Households. The economy is populated by a representative firm and by a countable set I of infinitely lived households indexed by $i \in I$. If I is finite, there is a finite set of types with a continuum of identical agents within each type. Households have identical additively separable preferences over sequences of consumption $c^i \equiv \{c^i(s^t)\}_{s^t \in S}$ of the form:

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c^i(s^t)) = E_0 \sum_{t=0}^{\infty} \beta^t u(c^i(s^t)) \quad (1)$$

where $\beta \in (0, 1)$ is the subjective discount factor and E_0 denotes the expectation conditional on information at date $t = 0$. The period utility function u is assumed to be strictly increasing, strictly concave and continuously differentiable, with $\lim_{c^i \rightarrow 0} u'(c^i) = \infty$, and $\lim_{c^i \rightarrow \infty} u'(c^i) = 0$.

At each date-state s^t , households can trade in physical capital to insure against uncertainty⁵. Capital depreciates at a constant rate δ , and its one period payoff net of depreciation is denoted by $r(s^{t+1})$ for each $s^{t+1}|s^t$. At each node, household $i \in I$ receives

⁵The present framework can be easily extended to the presence of trade in many assets. Further, the next section shows how these trading arrangements can be supported by a competitive financial intermediation sector.

asset income from his previous period investments $k^i(s^{t-1})$. In addition, he also receives a stochastic labour endowment $\epsilon^i(s^t)$, following a stationary Markov chain with S_ϵ values, where $0 < \epsilon^1 \dots < \epsilon^{S_\epsilon} < \infty$. Given this, his individual labor income is equal to $w(s^t)\epsilon^i(s^t)$, where $w(s^t)$ is the aggregate wage rate paid by the firm. Further, his date-state s^t budget constraint can be expressed as:

$$c^i(s^t) + k^i(s^t) = \omega^i(s^t) \quad (2)$$

$$\omega^i(s^{t+1}) \equiv w(s^{t+1})\epsilon^i(s^{t+1}) + r(s^{t+1})k^i(s^t) \quad (3)$$

where $\omega^i(s^t)$ represents the individual level of wealth. At period $t = 0$, the budget constraint takes the same form with $\omega^i(s^0) = w(s^0)\epsilon^i(s^0) + r(s^0)k^i(s^{-1})$, where the initial shock $\epsilon^i(s^0)$ and the initial capital holdings $k^i(s^{-1}) \in \mathbb{R}_+$ are given. Apart from the budget constraint, household $i \in I$ also faces a possibly endogenous and state-dependent trade restriction on capital holdings of the form:

$$k^i(s^t) \geq \underline{k}^i(s^t). \quad (4)$$

As to the previous trading restriction, we consider two different cases. In the first case, we assume that households cannot commit on the trading contracts, and the limits on capital holdings are endogenously determined at the level that prevents default in equilibrium. While these limits are simply imposed here, the next section shows that they would arise in equilibrium in the presence of a competitive intermediation sector. In the second case, we assume that the restrictions are of the form $k^i(s^t) \geq \underline{k}$, where \underline{k} is some exogenously fixed level that is chosen ad hoc, as usual in the incomplete markets literature (see Heaton and Lucas (1996), Marcet and Singleton (1999) or Telmer (1993)).

Production and Market Clearing. At each date-state s^t , the representative firm uses capital $K(s^t) \in \mathbb{R}_+$ and aggregate labor $L(s^t) \in (0, 1)$ to produce a single good $y(s^t) \in \mathbb{R}_+$ with the constant returns to scale technology:

$$y(s^t) = f(z(s^t), K(s^t), L(s^t)) \quad (5)$$

where $z(s^t)$ is an aggregate productivity shock that follows a stationary Markov chain with S_z values, where $0 < z^1 \dots < z^{S_z} < \infty$. To simplify notation, we denote total output including undepreciated capital by $F(z(s^t), K(s^t), L(s^t)) = f(z(s^t), K(s^t), L(s^t)) + (1 - \delta)K(s^t)$.

Each period, after observing the realization of the productivity shock, the firm rents capital and labor to maximize period profits:

$$F(z(s^t), K(s^t), L(s^t)) - w(s^t)L(s^t) - r(s^t)K(s^t) \quad (6)$$

leading to the following first order conditions:

$$w(s^t) = f_L(z(s^t), K(s^t), L(s^t)) \quad (7)$$

$$r(s^t) = f_K(z(s^t), K(s^t), L(s^t)) + 1 - \delta \quad (8)$$

Finally, labor and asset market clearing require that the sum over households of the labor income shocks and the capital asset holdings are equal to the total labor supply and to the aggregate capital stock respectively. Further, the good's market clearing condition requires that the sum of aggregate consumption and investment are equal to aggregate output.

Recursive Competitive Equilibrium. In the present model, the aggregate state of the economy is given by $S = (\Psi(\epsilon, k), z)$, where $\Psi(\epsilon, k)$ represents the joint distribution of consumers over individual capital holdings k and idiosyncratic productivity status ϵ . Note that, in the presence of only one asset, we can either include the individual wealth ω or the individual capital holdings k as a state variable, but we choose k for expositional simplicity. Further, introducing the aggregate capital K as a separate state becomes unnecessary, since it is just a particular moment of the wealth-productivity distribution.

The law of motion of the two shocks is exogenously given by a joint discrete Markov process with transition matrix $\Pi(\epsilon', z' | \epsilon, z)$, which can allow the two shocks to be correlated. Further, households perceive that Ψ evolves according to:

$$\Psi(\epsilon', k') = \Gamma[\Psi(\epsilon, k), z]$$

where Γ represents the stochastic mapping or transition function from the current aggregate state into tomorrow's wealth-productivity distribution. Finally, the individual state vector includes the individual labour productivity and the individual capital holdings (ϵ, k) . Given this, the relevant state variables for a household are summarized by the vector $(\epsilon, k; S) = (\epsilon, k; \Psi(\epsilon, k), z)$.

Definition 2.1: Given a vector of initial asset holdings $k_{-1} \equiv (k^i(s^{-1}))_{i \in I}$, a vector of initial shocks $(z_0, \epsilon_0) \equiv (z_0, (\epsilon_0^i)_{i \in I})$ and a transition matrix Π for the shocks, a *recursive competitive equilibrium* relative to the trading limits $\underline{k}(\epsilon; S)$ is defined by a law of motion Γ , a vector of factor prices $(r, w) = (r(S), w(S))$, value functions $W = W(\epsilon, k; S)$ and individual policy functions $(c, k') = (c(\epsilon, k; S), k(\epsilon, k; S))$ such that⁶:

- (i) *Utility Maximization:* For each $i \in I$, W and (c, k') solve the following problem given k^{-1} , (z_0, ϵ_0) , Π , Γ and (r, w) :

$$W(\epsilon, k; \Psi(\epsilon, k), z) = \max_{c, k'} \left\{ u(c) + \beta \sum_{\epsilon', z'} \Pi(\epsilon', z' | \epsilon, z) W(\epsilon', k'; \Psi(\epsilon', k'), z') \right\} \text{ s.t.}$$

$$c + k' = w(S)\epsilon + r(S)k$$

⁶Note that we have assumed that households have the same preferences and identical processes for the idiosyncratic shocks, implying that the policy rules as functions of the states are identical across them.

$$\Psi(\epsilon', k') = \Gamma[\Psi(\epsilon, k), z]$$

$$k' \geq \underline{k}(\epsilon'; \Psi(\epsilon', k'), z') \text{ for all } (\epsilon', z') | (\epsilon, z) \text{ with } \Pi(\epsilon', z' | \epsilon, z) > 0.$$

(ii) *Profit Maximization:* Factor prices satisfy the firm's optimality conditions, i.e., $w(S) = w(z, K) = f_L(z, K, L)$ and $r(S) = r(z, K) = f_K(z, K, L) + 1 - \delta$.

(iii) *Market Clearing:*

$$\begin{aligned} \int k(\epsilon, k; S) \Psi(\epsilon, k) d\epsilon dk &= K' \\ \int \epsilon \Psi(\epsilon, k) d\epsilon dk &= L \\ \int [c(\epsilon, k; S) + k(\epsilon, k; S)] \Psi(\epsilon, k) d\epsilon dk &= F(z, K, L) + (1 - \delta)K. \end{aligned}$$

(iv) *Consistency:* Γ is consistent with the agent's optimal decisions, in the sense that it is generated by the optimal decision rules and by the law of motion of the shocks.

As reflected in condition (i), we have imposed endogenous limits on the individual capital holdings that depend on the next period values of the two shocks and on the (perceived) next period aggregate wealth-productivity distribution. Whereas the exact specification and the determination of these limits will be given below, they will be chosen to guarantee that an agent only chooses levels of individual capital holdings such that he has no incentive to default in any continuation state with positive probability next period. Given this, the effective limit on capital holdings faced by a household will be given by the tightest of the state-dependent limits. In terms of our previous notation, this implies that:⁷

$$\underline{k}^i(s^t) \equiv \max_{\{\epsilon(s^{t+1}|s^t), z(s^{t+1}|s^t)\}} \underline{k}(\epsilon(s^{t+1}|s^t); \Psi[\epsilon(s^{t+1}|s^t), k(s^t)], z(s^{t+1}|s^t)).$$

Note also that the previous definition of equilibrium allows for the presence of a finite or an infinite number of households. For comparison with a significant portion of the incomplete markets literature, however (see e.g. Heaton and Lucas (1996), Alvarez and Jermann (2000) and Zhang (1997a, 1997b)), our benchmark model version assumes that there are two types of households and a continuum of identical households of each type. Further, as usual in the literature with two types, the individual labor income shocks are assumed to have no aggregate productivity effects and are therefore perfectly negatively correlated across types. In this sense, they are not truly idiosyncratic shocks but rather shocks to the distribution of individual labor income across the two types.

⁷Since the trading restriction has to be satisfied for all possible continuation states $(\epsilon', z') | (\epsilon, z)$, the effective limit faced by the households will not be a function of the current shocks if the probability of all future states is strictly positive. This is not the case, however, with our present calibration.

The previous assumptions have the following implications. First, the aggregate labor supply is constant and it can therefore be normalized to one without loss of generality. Second, the aggregate capital stock K and the vector (ϵ^i, k^i) for $i = 1$ or 2 provide a complete description of the wealth-productivity distribution, since we can infer (ϵ^{-i}, k^{-i}) from $\epsilon^{-i} = 1 - \epsilon^i$ and $k^{-i} = K - k^i$. The aggregate state vector for a household in the presence of two types is therefore given by $S = (z, K, \boldsymbol{\epsilon}, \mathbf{k})$, where $(K, \boldsymbol{\epsilon}, \mathbf{k})$ represents the wealth-productivity distribution, and households regard the variables in bold as beyond their control when making their individual decisions. Here, it is important to note that we distinguish (k, ϵ) and $(\mathbf{k}, \boldsymbol{\epsilon})$ only when posting the individual decisions, but we set $k = \mathbf{k}$ and $\epsilon = \boldsymbol{\epsilon}$ to impose the equilibrium. Note also that the law of motion Γ simplifies to:

$$(K', \boldsymbol{\epsilon}', \mathbf{k}') = \Gamma[(K, \boldsymbol{\epsilon}, \mathbf{k}), z].$$

In addition, the optimization problem in the previous definition of equilibrium can now be expressed as:

$$W(\epsilon, k; (K, \boldsymbol{\epsilon}, \mathbf{k}), z) = \max_{c, k'} \left\{ u(c) + \beta \sum_{(\epsilon', z')} \Pi(\epsilon', z' | \epsilon, z) W(\epsilon', k'; (K', \boldsymbol{\epsilon}', \mathbf{k}'), z') \right\} \text{ s.t.} \quad (9)$$

$$c + k' = w(S)\epsilon + r(S)k \quad (10)$$

$$(K', \boldsymbol{\epsilon}', \mathbf{k}') = \Gamma[(K, \boldsymbol{\epsilon}, \mathbf{k}), z] \quad (11)$$

$$k' \geq \underline{k}(\epsilon'; K', z') \text{ for all } (\epsilon', z') | (\epsilon, z) \text{ with } \Pi(\epsilon', z' | \epsilon, z) > 0, \quad (12)$$

whereas the market clearing conditions are given by:

$$k(\epsilon, k; S) + k(1 - \epsilon, K - k; S) = K' \quad (13)$$

$$L = 1 \quad (14)$$

$$c(\epsilon, k; S) + c(L - \epsilon, K - k; S) + K' = F(z, K, L) + (1 - \delta)K \quad (15)$$

As stated earlier, consistency requires that $k' = \mathbf{k}'$ and $\epsilon' = \boldsymbol{\epsilon}'$, hence we denote the reduced form or equilibrium policy and value functions by $W(\epsilon, k; K, z)$, $c(\epsilon, k; K, z)$ and $\underline{k}(\epsilon; K, z)$. In addition, note that the endogenous trading restriction with $I = 2$ only depends on the aggregate capital stock and on the two exogenous shocks. A detailed description of the determination of these limits is provided in what follows.

The Endogenous Trading Restriction. As stated earlier, our benchmark model version assumes that households cannot commit on the trading contracts. Further, we assume that a household will be seized from his asset holdings and from future asset trade forever upon default, implying that his only source of income from the default period on will be his labor income.⁸ The autarky value V can therefore be expressed recursively as:

$$V(\epsilon; (K, \epsilon, \mathbf{k}), z) = u(w(S)\epsilon) + \beta \sum_{(\epsilon', z')} \Pi(\epsilon', z' | \epsilon, z) V(\epsilon'; (K', \epsilon', \mathbf{k}'), z'). \quad (16)$$

Several remarks are worth noting. First, in contrast with some of the literature under complete markets and no commitment, such as Alvarez and Jermann (2000,2001), where the autarky value V is exogenous, the autarky value in the present framework is a function of the wealth-productivity distribution. As we will see later, this is due to the fact that the distribution influences the aggregate capital accumulation, which in turn affects future wages and therefore the future value of financial autarky. In particular, a higher dispersion of the wealth-productivity distribution, represented by a lower (negative) \mathbf{k} , leads to a higher capital accumulation and to higher future wages. Thus, in spite of the fact that individual capital holdings do not affect the value of autarky directly, we have that $V_k(\epsilon; (K, \epsilon, k), z) < 0$ in equilibrium, at least for low (negative) levels of k .

Second, as shown in the next section, default cannot be an equilibrium outcome in the presence of a competitive intermediation sector. In other words, the intermediation sector would set the borrowing limits such that no household has an incentive to default in equilibrium. On the other hand, we consider the loosest possible of such limits. In other words, we will analyze the economy with limits that are *not too tight*, in the sense that they are determined by the individual capital holdings \underline{k} at which the agent is indifferent between defaulting or paying back the debt, i.e.,

$$\underline{k}(\epsilon; K, z) = \{\underline{k} : W(\epsilon, \underline{k}; (K, \epsilon, \underline{k}), z) = V(\epsilon; (K, \epsilon, \underline{k}), z)\}. \quad (17)$$

Note that the previous limit has to be negative, since no agent would default with a positive level of asset holdings. In this case, he could clearly afford a higher current consumption than in autarky and at least as high of a life-time utility as in autarky from next period on. Finally, as we will see later, an increase in k leads to a higher trading value due to its positive first order effect on the individual level of wealth. Given this, the previous wealth restriction defines a unique limit for every date-state, and it implies that:

$$W(\epsilon, k; (K, \epsilon, k), z) \geq V(\epsilon; (K, \epsilon, k), z) \quad \forall k \geq \underline{k}(\epsilon; K, z). \quad (18)$$

As stated earlier, the allocations resulting under these limits will first be compared to the equilibrium allocations in the presence of fixed and exogenous limits on the capital

⁸This punishment for default closely resembles the bankruptcy procedures under Chapter 7.

holdings. In addition, for comparison with the literature with enforcement constraints (see e.g. Alvarez and Jermann (2000,2001) or Kehoe and Perri (2002)), they will also be compared to the equilibrium allocations when markets are complete but households cannot commit on the trading contracts. This framework is discussed in section four. This next section shows that the endogenous limits specified above would arise in equilibrium in the presence of a competitive intermediation sector.

3. THE INTERMEDIATION SECTOR

For this section, we assume that, at each node s^t , households can trade through financial intermediaries in a risky asset with an endogenous one period payoff of $R(s^{t+1})$ for each future state $s^{t+1}|s^t$. In particular, if a household invests in the asset ($a^i(s^t) > 0$), for each unit of consumption he or she gives up at node s^t , the intermediary promises to pay back $R(s^{t+1})$ units of the consumption good if date-state $s^{t+1}|s^t$ is realized. Further, if the household borrows ($a^i(s^t) < 0$), for each unit of consumption he or she receives, he or she promises to pay back $R(s^{t+1})$ units to the intermediary if the continuation state $s^{t+1}|s^t$ is realized. We assume that all intermediaries take the one period payoffs as given. Further, they cannot price discriminate, in the sense that they have to pay the same return to investors as they charge to borrowers⁹. Under these assumptions, the budget constraint and wealth accumulation equation of household $i \in I$ can be written as follows:

$$c^i(s^t) + a^i(s^t) = \omega^i(s^t)$$

$$\omega^i(s^{t+1}) \equiv w(s^{t+1})\epsilon^i(s^{t+1}) + R(s^{t+1})a^i(s^t).$$

The intermediaries live for two periods. At the beginning of the first period, they set limits on the agents' capital holdings by only allowing them to take asset positions that do not violate lower bounds on their asset levels at any possible continuation state. Further, the limits are set in such a way that no intermediary has an incentive to deviate from them. Households are therefore subject to a trading constraint of the form:

$$a^i(s^t) \geq \underline{a}^i(s^t).$$

The cash flows of the intermediaries can be described as follows. During the first period, they trade consumption goods with the households and collect a total amount of $A(s^t) \geq 0$ goods. Further, they transform this (or some portion of it) into physical capital $k(s^t) \leq A(s^t)$, which is rented to the representative firm. For simplicity, we assume that this transformation is one-to-one. In the second period, they receive rental income of $r(s^{t+1})k(s^t)$ from the firm

⁹This is the key assumption differentiating this framework from Dubey, Geanakoplos and Shubik (2005), where the authors allow for different "saving" and "borrowing" rates. Under their assumption, default can be an equilibrium outcome.

if state $s^{t+1}|s^t$ is realized, and they have to honor the trading contracts with the households by paying back $R(s^{t+1})A(s^t)$. We assume that all the intermediaries can commit to repay back their debt to the households at any possible contingency, but they cannot be forced to lend to or borrow from any household. Further, we assume that $A(s^t) > 0$, implying that intermediaries cannot be solely making pure arbitrage profits but have to mediate between the households and the production sector¹⁰.

We focus on symmetric equilibria where all intermediaries hold the same portfolio and households have no incentives to default, leading to the following restrictions. First, an equilibrium implies that $k(s^t) = K(s^t)$, where $K(s^t)$ is the demand for capital of the representative firm. Second, the following condition has to hold:

$$\sum_{s^{t+1}} \pi(s^{t+1} | s^t) r(s^{t+1}|s^t) \leq \sum_{s^{t+1}} \pi(s^{t+1} | s^t) R(s^{t+1}|s^t). \quad (19)$$

To see why this is the case, note that the intermediaries could otherwise make arbitrarily large profits by demanding arbitrarily large funds $A(s^t)$ and by setting $k(s^t) = A(s^t)$. Third, since the intermediary can commit to repay back its debt to the households at each node s^t , solvency requires that $R(s^{t+1})A(s^t) \leq r(s^{t+1})k(s^t)$ for all $s^{t+1}|s^t$. This, together with the fact that $k(s^t) \leq A(s^t)$, implies that $R(s^{t+1}) \leq r(s^{t+1})$ for all $s^{t+1}|s^t$. Finally, combining the last condition with (19), it becomes clear that the only possible equilibrium is to have $R(s^{t+1}) = r(s^{t+1})$ at all $s^{t+1}|s^t$ and $k(s^t) = A(s^t) = K(s^t)$ at all s^t . Thus, all intermediaries make zero profits.

Clearly, there are many allocations with different borrowing constraints that satisfy the above restrictions. In this paper, we consider the loosest a possible ones of such limits or in other words we define *limits that are not too tight*, in the sense that some agent would default under some continuation state if the limits were made looser by any $\varepsilon > 0$. Formally, the limits $\underline{a}^i(s^t)$ have to satisfy:

$$W(\epsilon^i(s^t), \underline{a}^i(s^t); \underline{S}(s^t)) = V(\epsilon^i(s^t); \underline{S}(s^t))$$

where $\underline{S}(s^t) = ((K(s^t), \underline{a}^i(s^t), \epsilon^i(s^t)), z(s^t))$ and the contract and autarky values W and V are defined as in the previous section. Note that we have just shown that agents face the same return on $\underline{a}^i(s^t)$ as on physical capital in the previous section. Given this, we can also use the same value functions. Further, in order to show that the previous limits actually arise in equilibrium, we have to show that no intermediary can make positive profits by loosening the trading limits relative to the limits that are not too tight. This is shown by the following proposition, which also establishes that there does not exist any symmetric equilibrium with limits that allow for default.

¹⁰This assumption is necessary to guarantee the existence of symmetric equilibria where all intermediaries hold the same portfolio.

Proposition 3.1 *The allocation with limits that are not too tight is a (symmetric) equilibrium. Further, a symmetric equilibrium with default does not exist.*

Proof. We first prove that the allocation with limits that are not too tight is a symmetric equilibrium. In particular, we consider a symmetric equilibrium with the loosest possible limits that avoid default in equilibrium, and we then show that no intermediary can achieve positive or zero profits by loosening these limits. To do this, assume without loss of generality that the participation constraint is binding for the agents of type one at some possible contingency \tilde{s} , implying that $W(\epsilon_1(\tilde{s}), a_1(\tilde{s}); S(\tilde{s})) = V(\epsilon_1(\tilde{s}), a_1(\tilde{s}); S(\tilde{s}))$. We now show that no intermediary can build a portfolio $\bar{A}(\tilde{s}) = \bar{a}_1(\tilde{s}) + \bar{a}_2(\tilde{s}) > 0$ which gives him at least zero profits and which involves more lending $\bar{a}_1(\tilde{s}) < a_1(\tilde{s})$ to the type one agents. To see this, note first that the intermediaries need to satisfy the following condition to be solvent at each future state $s^{t+1}|\tilde{s}$:

$$\bar{R}(s^{t+1})\bar{A}(\tilde{s}) \leq r(s^{t+1})\bar{k}(\tilde{s})$$

where $\bar{R}(s^{t+1})$ is the return offered by the intermediary at $s^{t+1}|\tilde{s}$ and $\bar{k}(\tilde{s})$ is the total capital rented to the firm at \tilde{s} . Second, since $\bar{k}(\tilde{s}) \leq \bar{A}(\tilde{s})$, it has to be the case that $\bar{R}(s^{t+1}) \leq r(s^{t+1}) = R(s^{t+1})$ for all $s^{t+1}|\tilde{s}$, where the last equality follows from the fact that R and r satisfy condition (19) for a symmetric equilibrium with no default discussed above. Finally, note that the agents of type one will default at some contingency $\hat{s}|\tilde{s}$ with $\pi(\hat{s}|\tilde{s}) > 0$. It therefore follows that $\bar{R}(\hat{s}) < r(\hat{s}) = R(\hat{s})$ for some $\hat{s}|\tilde{s}$ with $\pi(\hat{s}|\tilde{s}) > 0$, implying that this is a strictly worse deal for the type two agents. Given this, the intermediary will not be able to build this portfolio. To see that the last condition has to hold, note that the intermediary has at most $r(\hat{s})\bar{A}(\tilde{s})$ goods available at node $\hat{s}|\tilde{s}$. On the other hand, he has to pay out $\bar{R}(\hat{s}|\tilde{s})(\bar{A}(\tilde{s}) - \bar{a}_1(\tilde{s}))$ to the type two agents. Since the type one agents will only default if $\bar{a}_1(\tilde{s}) < 0$, however, this can only be done if $\bar{R}(\hat{s}) < r(\hat{s}) = R(\hat{s})$ at $\hat{s}|\tilde{s}$. Alternatively, the intermediary could decrease $\bar{R}(\hat{s})$ so as to make the type one agents not want to default any more, but this would directly imply that $\bar{R}(\hat{s}) < r(\hat{s}) = R(\hat{s})$ at $\hat{s}|\tilde{s}$. Type two agents will therefore not be willing to accept the deal in either case.

We now prove that symmetric equilibria with default cannot exist. To do this, assume first that the limits are such that, at a given current state \tilde{s} , there exists at least a future state \hat{s} (or more generally a set of states) where agents of the first type will default. We now show that this equilibrium cannot exist, since the intermediaries will be able to make positive profits by not “lending” to the types with positive default probabilities. To see this, we first note that the following condition has to hold in equilibrium:

$$\sum_{s^{t+1}|\tilde{s}} \pi(s^{t+1}|\tilde{s})r(s^{t+1}) \leq \sum_{s^{t+1}|\tilde{s}} \pi(s^{t+1}|\tilde{s})R(s^{t+1}) - \pi(\hat{s}|\tilde{s})R(\hat{s})\frac{a_1(\tilde{s})}{A(\tilde{s})} \quad (20)$$

where $a_1(\tilde{s})$ are the asset holdings of the type one agents and $A(\tilde{s})$ are the total funds collected

by the intermediaries. Further, r and R represent the prices in the symmetric equilibrium with default. Note that, if the previous condition was not satisfied, the intermediaries could set $k(s^t) = A(s^t)$ and make arbitrarily large profits, since their cash flows at $s^{t+1}|\tilde{s}$ would then be given by:

$$\sum_{s^{t+1} \neq \widehat{s}|\tilde{s}} \pi(s^{t+1}|\tilde{s}) [r(s^{t+1}) - R(s^{t+1})] A(\tilde{s}) + \pi(\widehat{s}|\tilde{s})r(\widehat{s})A(\tilde{s}) - \pi(\widehat{s}|\tilde{s})R(\widehat{s})a_2(\tilde{s}) > 0$$

Second, since $k(s^t) \leq A(s^t)$ at all nodes and the intermediaries have to be solvent at every possible contingency, the only equilibrium with $A(s^t) > 0$ is given when $k(s^t) = A(s^t)$ and (20) satisfied with equality. In this case, the intermediaries make zero profits. On the other hand, since the agents of type one do not want to default unless $a_1(\tilde{s}) < 0$, it also follows that:

$$\sum_{s^{t+1}|\tilde{s}} \pi(s^{t+1}|\tilde{s})r(s^{t+1}) > \sum_{s^{t+1}|\tilde{s}} \pi(s^{t+1}|\tilde{s})R(s^{t+1}) \quad (21)$$

As we see, condition (21) implies that an intermediary could make positive profits under the current prices by only accepting deposits from the type two agents and by not lending to the type one agents, i.e., by setting $a_1(\tilde{s}) = 0$. Under this profitable deviation, any intermediary offering the original contract would be driven out of the market. Further, there would not be any lending to the type one agents, contradicting the existence of an equilibrium with default of the type one agents. ■

Proposition 3.1. establishes the micro foundations for our borrowing limits, and it implies that the limits set by a competitive intermediation sector are such that no household has an incentive to default in a symmetric equilibrium. Further, among these equilibria, we chose the one with the loosest possible limits or with limits that are not too tight, since this allows for the highest possible risk sharing.

4. THE COMPLETE MARKETS MODEL

The present section discusses the model when markets are complete. To implement this allocation with sequential trading, we assume that households can trade in a complete set of one period ahead state contingent claims (or Arrow securities) through financial intermediaries that set the trading (borrowing) limits. Under these assumptions, the constraints faced by the households at node s^t are given by:

$$c^i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)a^i(s^{t+1}|s^t) \leq \omega^i(s^t) \quad (22)$$

$$\omega^i(s^{t+1}|s^t) = w(s^{t+1}|s^t)\epsilon^i(s^{t+1}|s^t) + a^i(s^{t+1}|s^t) \quad (23)$$

$$a^i(s^{t+1}|s^t) \geq \underline{a}^i(s^{t+1}) \quad (24)$$

where $a^i(s^{t+1}|s^t)$ represents the amount of state contingent claims chosen by household $i \in I$ at node s^t . Further, $q(s^{t+1}|s^t)$ denotes the price of one unit of consumption good delivered at $t + 1$ contingent on the realization $s^{t+1}|s^t$. As before, we consider the loosest possible limits on the Arrow security holdings that avoid default in equilibrium. In other words, the limits in (24) are assumed to be *not too tight*, in the sense that they satisfy the following condition:

$$W^{CM}(\epsilon^i(s^t), \underline{a}^i(s^t); \underline{S}(s^t)) = V^{CM}(\epsilon^i(s^t); \underline{S}(s^t)) \quad (25)$$

where $\underline{S}(s^t) = ((K(s^t), \epsilon^i(s^t), \underline{a}^i(s^t)), z(s^t))$, and W^{CM} and V^{CM} represent the trading and autarky value functions under complete markets, defined as in the incomplete market economies of section 2.

Each period, after observing the realization of the productivity shock, the firm rents labor from the households and physical capital from the intermediaries to maximize the period profits. On the other hand, the intermediary purchases capital $k(s^{t+1}|s^t)$ and rents it to the firm, earning a rental revenue of $r(s^{t+1}|s^t)k(s^{t+1}|s^t)$ and a liquidation value of $(1 - \delta)k(s^{t+1}|s^t)$ the following period if node $s^{t+1}|s^t$ is realized. Further, to finance the capital purchases, they sell the future consumption goods in the spot markets for one period ahead contingent claims. Given this, the zero profit condition implied by the presence of perfect competition in the intermediary sector requires that:

$$1 = \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)[r(s^{t+1}|s^t) + (1 - \delta)]. \quad (26)$$

Note that, for the state contingent debt issued by the intermediary to match the demand from the households it must be the case that:

$$\sum_i a^i(s^{t+1}|s^t) = [r(s^{t+1}|s^t) + (1 - \delta)]k(s^{t+1}|s^t). \quad (27)$$

Finally, the resource constraint of this economy is given by:

$$\sum_{i \in I} c^i(s^t) + K(s^{t+1}|s^t) = F(z(s^t), K(s^t), L(s^t)) + (1 - \delta)K(s^t). \quad (28)$$

Consider an allocation satisfying (22)-(28). In spite of the fact that markets are complete, it is important to note that this allocation is not constrained efficient. To see why this is the case, note that the constrained efficient allocations of the previous economy can be calculated by solving the following central planning problem:

$$\begin{aligned} & \text{Max}_{\{c^i, K\}} \sum_{i \in I} \alpha^i \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c^i(s^t)) \quad \text{s.t.} \\ & \sum_{i \in I} c^i(s^t) + K(s^{t+1}) = F(z(s^t), K(s^t), L(s^t)) + (1 - \delta)K(s^t) \end{aligned} \quad (29)$$

$$\sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \pi(s^r) u(c_i(s^r)) \geq V^P(s^t) \quad (30)$$

where α_i is the initial Pareto weight assigned by the planner to each household. Further, $V^P(s^t) \equiv V^P(\epsilon^i(s^t); K(s^t), z(s^t))$ represents the autarky or outside option value assigned by the planner to household $i \in I$. Clearly, standard dynamic programming is inapplicable to the present framework, since future decision variables appear in the enforcement constraints. On the other hand, following Marcet and Marimon (1999), we can expand the state space with a pseudo state variable to formulate the problem recursively. In particular, if $\beta^t \gamma^i(s^t)$ is the Lagrange multiplier of the time t participation constraint of household i , and $\mu^i(s^{t-1})$ denotes the period t pseudo state variable of the household, the previous optimization problem can be transformed into the following one:

$$\inf_{\{\gamma^i\}} \sup_{\{c^i, K\}} H \equiv \sum_{i \in I} \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t \{u(c^i(s^t)) (\mu^i(s^t) + \alpha^i) - \gamma^i(s^t) V^P(s^t)\}$$

subject to the resource constraint in (29) and to the law of motion of $\mu^i(s^t)$, defined recursively by:

$$\mu^i(s^t) = \mu^i(s^{t-1}) + \gamma^i(s^t), \mu^i(s^{-1}) = 0 \text{ for } i = 1, 2. \quad (31)$$

It is easy to see that the solution to the previous problem can be characterized by the resource and participation constraints in (29)-(31) and by the following first order conditions:

$$\frac{u'(c^1(s^t))}{u'(c^2(s^t))} = \lambda(s^t) = \frac{(1 + v^2(s^t))}{(1 + v^1(s^t))} \lambda(s^{t-1}) \quad (32)$$

$$1 = \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \frac{u'(c^i(s^{t+1}))}{u'(c^i(s^t))} (1 + v^i(s^{t+1})) [\alpha z(s^{t+1}) K(s^{t+1})^{\alpha-1} + (1 - \delta)] \right\} \quad (33)$$

$$- \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \sum_{j=1,2} \frac{v^j(s^{t+1})}{u'(c^j(s^t))} V_K^{Pj}(\epsilon^j(s^{t+1}); K(s^{t+1}), z(s^{t+1})) \right\}$$

The terms $V_K^{Pj}(\epsilon^j(s^{t+1}); K(s^{t+1}), z(s^{t+1}))$ for $j = 1, 2$ on the right hand side of the previous equation represent the derivatives of the outside option value V^P with respect to the aggregate capital stock K . Further, we have expressed the equations in terms of the normalized multipliers λ and v_i , which simplify the system of equations and are given below:

$$v^i(s^t) = \frac{\gamma^i(s^t)}{\mu^i(s^{t-1}) + \alpha^i} \text{ for } i = 1, 2 \quad (34)$$

$$\lambda(s^t) = \frac{\mu^2(s^t) + \alpha^2}{\mu^1(s^t) + \alpha^1}, \text{ with } \lambda(s^{-1}) = \frac{\alpha^2}{\alpha^1}. \quad (35)$$

As shown by [Ábrahám and Cárceles-Poveda \(2005\)](#), the presence of the autarky effects $V_K^{P'}$ in equation (33) implies that the optimal central planning problem cannot be decentralized as above without imposing an upper bound on capital holdings¹¹. Further, we also show there that (i) an allocation that satisfies (29)-(32) and (34)-(35) together with the following equation:

$$1 = \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \frac{c^i(s^{t+1})^{-\sigma}}{c^i(s^t)^{-\sigma}} (1 + v^i(s^{t+1})) [\alpha z(s^{t+1}) K(s^{t+1})^{\alpha-1} + (1 - \delta)] \right\} \quad (36)$$

also satisfies equations (22)-(28) and vice versa. In addition, (ii) the autarky effects in the Euler equation of the planner are quantitatively unimportant, whereas (iii) the financial intermediaries cannot make profitable deviations by further loosening the trading limits that are not too tight. Given this and the fact that we do not impose any upper bounds on the capital holdings in the incomplete market economies of section 2, we will analyze the allocations satisfying (29)-(32), (34)-(35) and (36). Finally, the above equivalence result implies that the beginning of period relative Pareto weight of the two agents $\lambda(s^{t-1})$ corresponds to the initial wealth distribution $(a^i(s^t), a^{-i}(s^t))$ of the households in the decentralized equilibrium allocation. In this sense, the multiplier λ plays a very similar role to the vector $(k^i(s^{t-1}), k^{-i}(s^{t-1}))$ in the incomplete market allocations, which determines the distribution of consumption in current and future periods. We will use this property when comparing the complete and incomplete market allocations in section 6.

5. SOLUTION METHOD

The present section provides a detailed description of the algorithm used to solve for the equilibrium allocations of the previous economies. To find the solution to the incomplete markets economy with endogenous trading limits, we use a policy function iteration algorithm that is modified to include an endogenous and not rectangular grid for the states. It is important to note that the system of equations (9)-(15) could also be solved with a value function iteration algorithm. This solution, however, would involve an extended state space including the variables that agents regard as beyond their control as well as iterations on Γ to satisfy the consistency requirements. Since the method of policy function iterations imposes the key equilibrium conditions and uses directly the reduced policy and value functions, we opt for using this method.

The implementation can be described as follows. First, we define a grid on the endogenous state space, given by the individual level of capital k and by the aggregate capital stock K . Note that the grid on the exogenous state space (ϵ, z) is implicitly defined by our Markov assumption. Second, we fix some initial limits $\underline{k}(\epsilon; K, z)$. Given the grid on k , K and

¹¹See also [Kehoe and Perri \(2004\)](#) for a formal proof of this statement in a related model.

(ϵ, z) , our procedure finds then continuous equilibrium policy functions for the individual consumptions $c = c(\epsilon, k; K, z)$, the individual asset holdings $k' = k(\epsilon, k; K, z)$, and the law of motion for the aggregate capital stock $K' = K(\epsilon, k; K, z)$, such that all the conditions of the recursive competitive equilibrium defined earlier are satisfied for the given set of limits, $\underline{k}(\epsilon; K, z)$. Here, we have to make sure that the following Euler equation coming from the household's optimization problem is satisfied:

$$u'(c) \geq \beta \sum_{(\epsilon', z')} \Pi(\epsilon', z' | \epsilon, z) u'(c') [f_K(z', K', L') + 1 - \delta].$$

As usual, the above condition holds with equality only if the trading constraint is not binding, i.e., only if $k' > \underline{k}(\epsilon; K, z)$ for all $(\epsilon', z') | (\epsilon, z)$ such that $\Pi(\epsilon', z' | \epsilon, z) > 0$. Third, using the equilibrium policy functions, the value functions $W = W(\epsilon, k; K, z)$ and $V = V(\epsilon, k; K, z)$ are calculated recursively. Finally, the endogenous limits are loosened if $W(\epsilon, \underline{k}(\epsilon; K, z); K, z) > W(\epsilon, \underline{k}(\epsilon; K, z); K, z)$ and they are tightened otherwise. All the previous objects are approximated with continuous functions using linear interpolation over the finite and endogenous grid, and the procedure is repeated until convergence.

More precisely, given a set of endogenous limits \underline{k} , let h be the vector consisting of the policy functions of interest, i.e., $h = [c, k', K']$. Further, let T be a non-linear operator such that $T[h \ W \ V \ \underline{k}]$ satisfies the equilibrium system of equations and the participation constraints determining the limits. The solution to our problem is then a fixed point of T , i.e., a vector $[h \ W \ V \ \underline{k}]$ such that $[h \ W \ V \ \underline{k}] = T[h \ W \ V \ \underline{k}]$. To approximate the fixed point, we follow the steps below.

Step 1: Guess an initial vector $[h_0 \ W_0 \ V_0 \ \underline{k}_0]$, where $h_0 = [c_0, k'_0, K'_0]$.

Step 2: For each iteration $n \geq 1$, use the previous guess $[h_{n-1} \ W_{n-1} \ V_{n-1}]$ and \underline{k}_{n-1} to compute the new vector $[h_n \ W_n \ V_n]$ that satisfies the equilibrium conditions. Further, use the new vector $[h_n \ W_n \ V_n]$ to find the new lower bound \underline{k}_n such that $W_n(\epsilon, \underline{k}_n; K_n, z) \approx V_n(\epsilon, \underline{k}_n; K_n, z)$, and update the grid accordingly.

Step 3: Use $[h_n \ W_n \ V_n]$ and \underline{k}_n as the next initial guess and iterate until $[h_n \ W_n \ V_n \ \underline{k}_n]$ converges.

As reflected by the previous algorithm, solving this incomplete markets model with endogenous trading limits involves several computational difficulties. First, our state space is endogenous, a problem that we address by incorporating an additional fixed point problem, apart from the one used to find the policy functions, to find the state-dependent limits on the individual capital holdings. This also implies that our policy functions have to be calculated over a non-rectangular grid. Second, given the typical non-linearities of the policy and value

functions around the limits, and given that the limits in our model are endogenously determined at the level where the value function from staying in the contract is at least as large as the autarky value, it becomes clear that a good approximation of the value functions close to the limits is needed to obtain reliable results. To address this issue, we use a relatively high number of grid points and allow the limits to take values between grid points as well.

Obviously, the incomplete markets allocation with ad hoc fixed limits is solved with a version of the procedure that does not update the limits. Finally, to solve for the complete markets allocation, we use a version of the algorithm described above where the state space is given by $(\lambda, \epsilon; z, K)$ and where we iterate on the vector $[h \ W \ V]$, with $h = [c, \lambda', K']$.

6. QUANTITATIVE RESULTS

The present section discusses the quantitative results obtained for the benchmark model, which is calibrated following the asset pricing and real business cycle literature. The time period is assumed to be one quarter, and the discount factor and depreciation rate are therefore set to $\beta = 0.99$ and $\delta = 0.025$. Concerning the functional forms, we assume that the production function is Cobb Douglas, with a constant capital share of $\alpha = 0.36$. Further, the utility function of the households is assumed to be $u(c) = \log(c)$. Finally, the exogenous shock processes are assumed to be independent. In particular, the aggregate technology shock follows a two state Markov chain with $z \in \{z_l, z_h\} = \{0.99, 1.01\}$, and its transition matrix is given by:

$$\Pi_z = \begin{bmatrix} \pi_{ll} & \pi_{lh} \\ \pi_{hl} & \pi_{hh} \end{bmatrix} = \begin{bmatrix} 0.875 & 0.125 \\ 0.125 & 0.875 \end{bmatrix}.$$

As to the idiosyncratic income process, it is assumed to follow a seven state Markov chain. Further, the values and transition matrix are obtained by using the Hussey and Tauchen (1991) procedure to discretize the following process:

$$\epsilon^{i'} = (1 - \psi_\epsilon)\mu_\epsilon + \psi_\epsilon\epsilon^i + u, \quad u \sim N(0, \sigma_u^2).$$

where the shock parameters are set to $\psi_\epsilon = 0.956$ and $\sigma_u^2 = 0.082$, corresponding to quarterly adjusted estimates from annual data used by Aiyagari (1994). Since $\epsilon^{-i} = 1 - \epsilon^i$ and the values for ϵ are chosen to be symmetric around 0.5, the idiosyncratic productivity of the two types follows the same process and is perfectly negatively correlated across them.

6.1. Characterization of the Endogenous Limits. Before presenting our numerical results, we provide an approximate analytical characterization of the behavior of the equilibrium endogenous limits using the reduced form value functions. For expositional convenience, we assume for these analytical derivations that both the aggregate shock z and

the idiosyncratic shock of the first type ϵ follow a continuous $AR(1)$ process:¹²

$$z_{t+1} = (1 - \rho_z) + \rho_z z_t + \varepsilon_{zt+1} \text{ with } \varepsilon_{zt+1} \sim N(0, \sigma_z^2)$$

and

$$\epsilon_{t+1} = (1 - \rho_\epsilon)0.5 + \rho_\epsilon \epsilon_t + \varepsilon_{\epsilon t+1} \text{ with } \varepsilon_{\epsilon t+1} \sim N(0, \sigma_\epsilon^2).$$

Note that this assumption will allow us to express the differential effect of a change in the shocks on the equilibrium limits. Throughout the section, we denote the period t state vector of a household by $s_t = (k, \epsilon, z, K)$, the period t policy functions by $k' = g^k(s_t)$ and $c = g^c(s_t)$, and the law of motion of aggregate capital by $K = g^K(s_t)$. Further, we let $s^{t+1}|s^t = (g^k(s_t), \rho_\epsilon \epsilon + \varepsilon_{\epsilon t+1}, \rho_z z + \varepsilon_{zt+1}, g^K(s_t))$. Using this notation, the period t value functions for trade and autarky can be written as:

$$W(s_t) = u(g^c(s_t)) + \beta \int_{\varepsilon_\epsilon} \int_{\varepsilon_z} W(s_{t+1}) d\varepsilon_\epsilon d\varepsilon_z$$

$$V(s_t) = u(w(K, z)\epsilon) + \beta \int_{\varepsilon_\epsilon} \int_{\varepsilon_z} V(s_{t+1}) d\varepsilon_\epsilon d\varepsilon_z$$

As shown earlier, the endogenous trading restriction is determined by the following condition:

$$W(\underline{k}(\epsilon, z, K), \epsilon, z, K) = V(\underline{k}(\epsilon, z, K), \epsilon, z, K)$$

Thus, we can differentiate the previous equation to express the effects of a change in K , ϵ or z on the trading limits as follows:

$$\frac{\partial \underline{k}(\epsilon, z, K)}{\partial K} = -\frac{W_K(k, \epsilon; z, K) - V_K(k, \epsilon; z, K)}{W_k(k, \epsilon; z, K) - V_k(k, \epsilon; z, K)} \quad (37)$$

$$\frac{\partial \underline{k}(\epsilon, z, K)}{\partial \epsilon} = -\frac{W_\epsilon(k, \epsilon; z, K) - V_\epsilon(k, \epsilon; z, K)}{W_k(k, \epsilon; z, K) - V_k(k, \epsilon; z, K)} \quad (38)$$

$$\frac{\partial \underline{k}(\epsilon, z, K)}{\partial z} = -\frac{W_z(k, \epsilon; z, K) - V_z(k, \epsilon; z, K)}{W_k(k, \epsilon; z, K) - V_k(k, \epsilon; z, K)} \quad (39)$$

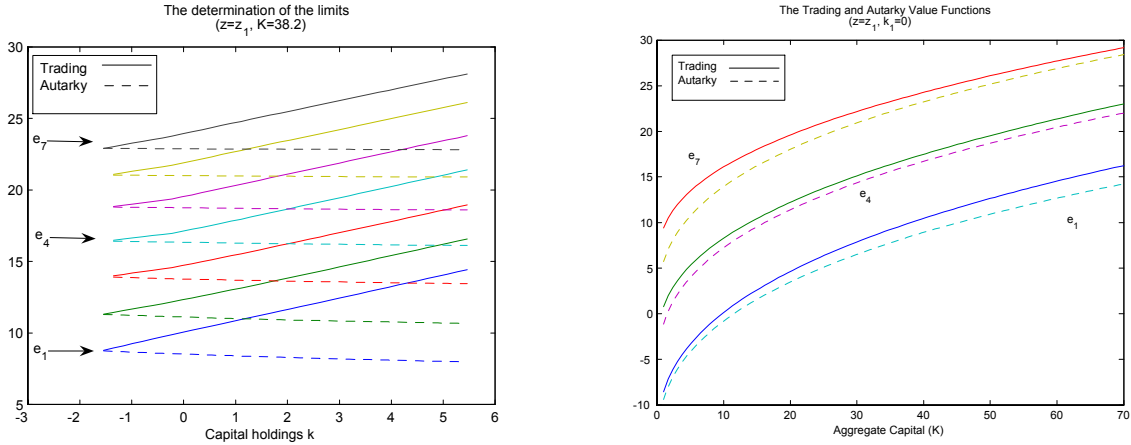
Looking at the previous equations, we can see that the sign of the derivatives of the limit crucially depends on the sign of $W_j(k, \epsilon; z, K) - V_j(k, \epsilon; z, K)$, where $V_j(k, \epsilon; z, K)$ and $W_j(k, \epsilon; z, K)$ represent the derivatives of the autarky and trading values with respect to the corresponding state variable $j \in \{k, \epsilon, z, K\}$. The two value functions are displayed on Figure 1 for different values of the state vector¹³. The left panel of Figure 1 depicts the two value functions for a medium level of aggregate capital inside the stationary distribution and

¹²More precisely, we need to assume that they both follow a truncated $AR(1)$ process to avoid negative values, while the truncation does not modify any of our results.

¹³Unless otherwise specified, all the graphs along the section set the technology shock to z_l . Further, the qualitative features remain unchanged if we condition on z_h .

for different values of ϵ as a function of the individual capital holdings k . As we see, for a given state vector $(\epsilon; z, K)$, the value of the trading arrangement increases in k , implying that $W_k(k, \epsilon; z, K) > 0$. On the other hand, the autarky value decreases in k for negative values of the initial capital holdings, implying that $V_k(k, \epsilon; z, K) < 0$ if $k < 0$. Given this, there exists a unique endogenous limit $\underline{k}(\epsilon; z, K)$ satisfying $W(\underline{k}(\epsilon; z, K), \epsilon; z, K) = V(\underline{k}(\epsilon; z, K), \epsilon; z, K)$, as reflected on the graph. Further, this also implies that the denominator of equations (37)-(39) is clearly positive.

Figure 1: Trading and Autarky Values



The right panel of Figure 1 depicts the trading and autarky values for different values of K and ϵ_1 when the individual capital holdings are set to $k_1 = 0$, in which case type 2 holds the entire aggregate capital stock. Throughout the section, we will frequently use this (initial) wealth distribution for two reasons. First, since our aim is to characterize the endogenous trading limits, it makes more sense to study an unequal distribution of wealth where the agents are not so far from being borrowing constrained. Second, this choice of the initial distribution, in our environment, corresponds to the empirical property of wealth distributions that there are many households concentrated around zero wealth and the rest of the economy owns all the productive assets.

As we see, both the autarky and trading values are increasing in the idiosyncratic income (productivity) ϵ and the aggregate capital K . In addition, results not depicted here show that both functions are also increasing in the aggregate technology shock z . Given this, the sign of the derivatives of the limits with respect to $j \in \{\epsilon, z, K\}$ entirely depends on the relative magnitude of the effects of a change in the corresponding state variable on the two value functions. In other words, if a marginal change in $j \in \{\epsilon, z, K\}$ increases more (less) the value of the trading arrangement than that of autarky, the borrowing limit will become looser (tighter) with j .

To get some intuition for the previous findings and to understand the signs of the numerators in (37)-(39), we can first express the derivatives of the value functions with respect to k as infinite sums. As shown in the appendix, they are given by:

$$\begin{aligned}
W_k(s_t) = & u'(c_t)(1 + r(K, z) - \delta) + \\
& + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau) \epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K} G^k(s_{\tau-1}) \\
& + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau) k_\tau \frac{\partial r(K_\tau, z_\tau)}{\partial K} G^k(s_{\tau-1})
\end{aligned} \tag{40}$$

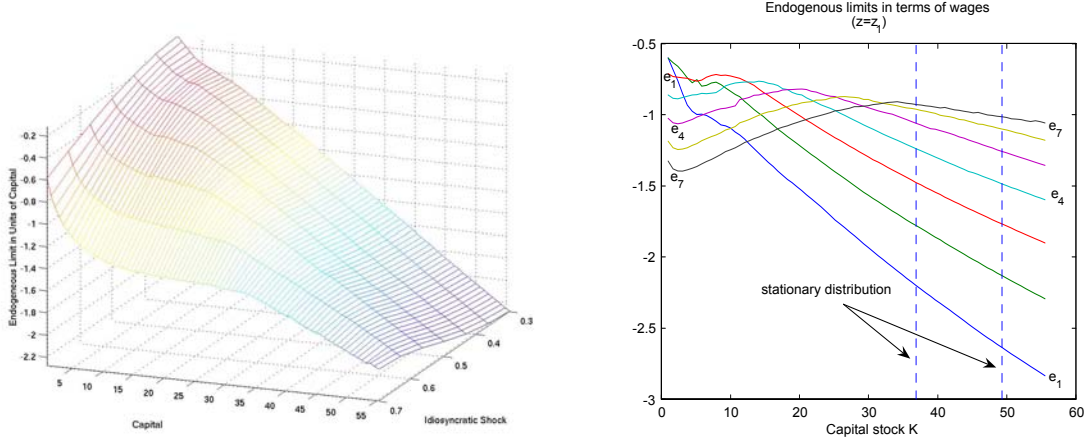
$$V_k(s_t) = E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} \epsilon_\tau u'(c_\tau^{au}) \frac{\partial w(K_\tau, z_\tau)}{\partial K} G^k(s_{\tau-1}) \tag{41}$$

where $c_\tau^{au} = \epsilon_\tau w(K_\tau, z_\tau)$ and $c_\tau = \epsilon_\tau w(K_\tau, z_\tau) + (r(K_\tau, z_\tau) + 1 - \delta)k_\tau - k_{\tau+1}$ represent the consumptions in autarky and in the trading arrangement respectively. Further, the terms $G^k(s_{\tau-1})$ capture the second order effects of a change in k through its effect on the evolution of the aggregate capital stock K . As shown in the Appendix, these last terms consist of products of the policy function derivatives g_k^k , g_K^k , g_k^K and g_K^K . Further, the definition of $G^k(s_\tau)$ suggests that, for negative values of k_τ , the second order general equilibrium effect is negative. In other words, a lower k will lead to an increase in the aggregate capital stock K when markets are incomplete. The key reason for this result is that a more disperse wealth distribution means that low wealth agents are closer to their borrowing limit. Given this, market clearing requires a downwards adjustment of the return on capital so that high wealth agents do not want to save so much. Further, this can only be achieved by increasing aggregate capital accumulation. Conversely, since an increase in k is equivalent to a reduction in the wealth dispersion, this clearly leads to a smaller capital accumulation. In addition, since the wage rate is increasing in capital, this also implies that $V_k(k, \epsilon; z, K) < 0$, confirming the findings above.

On the other hand, equation (40) reflects that an increase in k has three effects on the trading value. The first is a direct and positive effect due to the fact that a higher k increases the current individual wealth. The two latter terms are second order negative effects due to a change in the factor prices through the change in K . As reflected by the previous graph, however, the first order positive effect dominates, and $W_k(k, \epsilon; z, K)$ is therefore positive. Using similar arguments and the analytical expressions derived in the Appendix, it is easy to show that $W_j > 0$ and $V_j > 0$ for $j \in \{\epsilon, z, K\}$. Given this, the signs of the derivatives of the limits with respect to $j \in \{\epsilon, z, K\}$ entirely depend on the signs of $W_j(k, \epsilon; z, K) - V_j(k, \epsilon; z, K)$.

The endogenous limits $\underline{k}(\epsilon; z, K)$ on the total amount of borrowing are depicted on Figure 2 for different values of aggregate capital K and idiosyncratic income ϵ .

Figure 2: Endogenous Limits

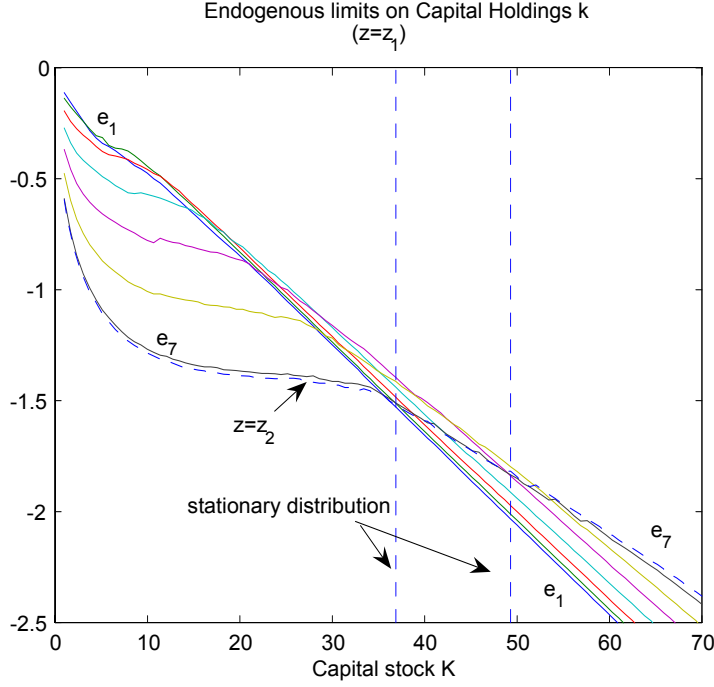


The left panel displays the value of the limits as a function of K and ϵ , whereas the right hand side figure depicts their value in terms of the individual labor incomes, that is, $\underline{k}(\epsilon; z, K) / (w(z, K)\epsilon)$. Several important facts are worth noting. First, the endogenous trading limits on the total amount of borrowing do vary to a great extent with the aggregate state of the economy. As we see on the right hand-side figure, households can borrow approximately between 90 and 250 percent of their individual labor incomes in the stationary distribution. Thus, the endogenous limits on physical capital that prevent default in equilibrium are significantly different from the ad hoc zero limit usually assumed in the incomplete markets literature. In addition, since we have a quarterly model, these results imply that households can borrow proportions of 22 to 63 percent of their annual income, which are in the ballpark of the amounts that U.S. individuals can safely borrow with multiple credit cards. In order to see better how the limits depend on the aggregate capital stock and on the idiosyncratic income separately, we have displayed the two dimensional cuts of the left panel of Figure 2 on Figures 3 and 4.

Figure 3 depicts the limits on the total amount of borrowing for different values of K and ϵ . Further, the dashed line at the bottom of the graph represents the limits for the high income shock and the high technology shock. As we see, the limits become looser with a higher aggregate capital stock for all levels of the idiosyncratic shock. To get more intuition for this result, we can look at the expressions for the numerator of (37), which is derived in the Appendix and is given by:

$$\begin{aligned}
 W_K(s_t) - V_K(s_t) &= E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \epsilon_{\tau} \frac{\partial w(K_{\tau}, z_{\tau})}{\partial K} G^K(s_{\tau-1}) (u'(c_{\tau}) - u'(c_{\tau}^{au})) \\
 &+ E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} u'(c_{\tau}) k_{\tau} \frac{\partial r(K_{\tau}, z_{\tau})}{\partial K} G^K(s_{\tau-1}).
 \end{aligned} \tag{42}$$

Figure 3: Endogenous Limits as a Function of Aggregate Capital



As before, the term G^K captures the second order effects of a change in the aggregate capital stock K through its effect on the future aggregate capital, and its sign is expected to be positive, as shown in the Appendix. If we consider an increase in the aggregate capital stock, equation (42) reflects that this has two different effects. First, it increases the present and future labor income in both the trading arrangement and in autarky. This is reflected by the first term on the right hand side of the equation, which is positive on average due to the fact $u'(c_\tau) - u'(c_\tau^{au}) > 0$, at least for time periods that are close to t . Note that the last inequality has to hold, since a household will only want to default at period t if $k_{t+1} - r_t k_t < 0$. Otherwise, he can enjoy a higher consumption than in autarky today and can still default next period. Consequently, we have $c_t < c_t^{au}$, and we also expect $c_\tau < c_\tau^{au}$ for some more periods τ after t due to the persistence of the income and therefore of the consumption processes. Note that this implies that a wage increase is expected to have a higher impact on the trading value than on the autarky value. Second, a higher aggregate capital generates a lower present and expected future rental price, reducing the asset liabilities and increasing the trading value above the autarky value. This is reflected by the second term on the right hand side of the equation, and the effect is also positive on average due to the fact that k_τ is negative at least for the initial periods. Given this, we have that $W_K(s_t) - V_K(s_t) > 0$ and $\frac{\partial k(\epsilon, z, K)}{\partial K} < 0$, as reflected by the previous graphs.

The dashed line on Figure 3 also shows that the limit is not monotone in the aggregate shocks. The effects of an increase in the aggregate technology shock can be analyzed using equation (43), which is derived in the appendix:

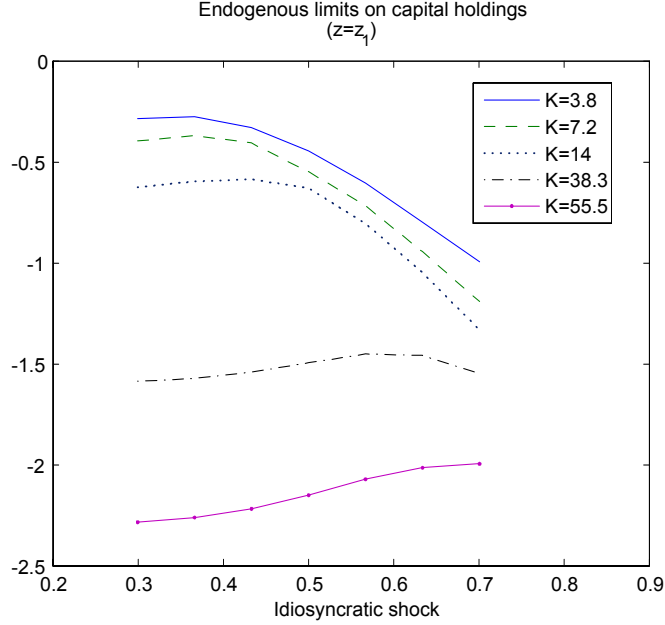
$$\begin{aligned}
W_z(s_t) - V_z(s_t) &= E_t \sum_{\tau=t}^{\infty} \left((\beta \rho_z)^{\tau-t} \epsilon_{\tau} \frac{\partial w(K, z)}{\partial z} (u'(c_{\tau}) - u'(c_{\tau}^{au})) \right) \\
&+ E_t \sum_{\tau=t+1}^{\infty} (\beta \rho_z)^{\tau-t} u'(c_{\tau}) k_{\tau} \frac{\partial r(K_{\tau}, z_{\tau})}{\partial z} \\
&+ E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} (u'(c_{\tau}) - u'(c_{\tau}^{au})) \epsilon_{\tau} \frac{\partial w(K_{\tau}, z_{\tau})}{\partial K} G^z(s_{\tau-1}) \\
&+ E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_{\tau}) k_{\tau} \frac{\partial r(K_{\tau}, z_{\tau})}{\partial K} G^z(s_{\tau-1}).
\end{aligned} \tag{43}$$

As before, the term G^z captures the second order effects of a change in z through its effect on the future aggregate capital, and its sign is expected to be positive. The previous equation reflects that a positive aggregate productivity shock has four effects. The first term captures the direct effect of an increase in z , leading to a higher present and future labor income in both the trading arrangement and autarky. This effect is positive on average. The second term is negative on average, and it captures the direct effect of an increase in z on the rental price, which increases with the aggregate shock, leading to higher asset liabilities in the trading arrangement. Finally, the third and fourth terms are second order effects. The first is positive on average, and it captures the fact that an increase in z leads to a higher aggregate capital and to higher future wages. The second is also positive on average, and it captures the fact that an increase in z will lead to a higher capital and to lower present and future asset liabilities. Thus, an increase in z leads to positive second order effects and to positive and negative first order effects. Suppose first that capital is low. In this case, the second-order positive effects are very high, and we expect $W_z(s_t) - V_z(s_t)$ to be positive. Further, the reverse behavior is expected when capital is very high. In other words, an increase in z will generate looser limits when capital is low and tighter limits when capital is high, as we see on the graph. Note that this result also suggests that borrowing constraints may get looser during booms in countries with low capital and get tighter in countries with high capital. Interestingly, the limits happen to be practically independent of the aggregate productivity shock in the stationary distribution of capital.

Finally, Figure 2 reflects that the endogenous limits are not monotone in the idiosyncratic shocks either. In particular, when capital is very scarce and its return is high, the limits are getting looser with the income shock, whereas the reverse is true when capital is high and its return is relatively low. This can be clearly seen on Figure 4, depicting the limits on total borrowing as a function of the idiosyncratic shock for different values of the aggregate

capital stock. As we see, the limits have a hump-shape pattern for intermediate levels of capital, since they become tighter (looser) for low to medium (medium to high) values of the idiosyncratic shock.

Figure 4: Endogenous limits on total borrowing



In order to understand the previous non-monotonic patterns, we consider an increase in the labor income shock and study its effect on the numerator of (38), which is derived in the appendix and given by:

$$\begin{aligned}
W_\epsilon(s_t) - V_\epsilon(s_t) &= E_t \sum_{\tau=t}^{\infty} (\beta \rho_\epsilon)^{\tau-t} w(K, z) (u'(c_\tau) - u'(c_\tau^{au})) \\
&\quad + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau) k_\tau \frac{\partial r(K_\tau, z_\tau)}{\partial K} G^\epsilon(s_{\tau-1}) \\
&\quad + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} (u'(c_\tau) - u'(c_\tau^{au})) \epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K} G^\epsilon(s_{\tau-1}).
\end{aligned} \tag{44}$$

As before, G^ϵ captures the second order effects of a change in ϵ through its effect on the future capital, and its sign is expected to be negative. To understand this, note that we have the same mechanism as in the case of G^k , in the sense that an increase in ϵ is equivalent to a reduction in the income dispersion, which leads to less capital accumulation. As shown by equation (44), an increase in ϵ has several effects. On the one hand, a higher shock increases

the labor income in the autarky value. This direct effect is reflected by the negative part of the first term $(-w(K, z)u'(c_\tau^{au}))$, which we label as complete markets (CM) effect. Note that, under complete markets, a change in the labor income shock would leave the consumption in the trading arrangement unaffected. Thus, a household with a higher income shock would have a higher incentive to default and would therefore be subject to a tighter limit. On the other hand, a higher income shock does lead to a consumption change in the trading arrangement when markets are incomplete. This effect is reflected by the positive part of the first term $(w(K, z)(u'(c_\tau)))$, which we denote as incomplete markets (IM) effect. The sum of these two direct effects is expected to be positive on average. Finally, a higher income shock leads to a change in the two factor prices through its effect on the future capital stock. This is what we call the capital accumulation effect (CA), arising due to the presence of a production sector. In particular, a higher income shock leads to a lower aggregate capital and to higher interest payments that decrease the trading value. This negative effect is reflected by the second term of equation (44). In addition, it also leads to lower wages in both scenarios. This effect is represented by the third term of (44), which is also negative on average.

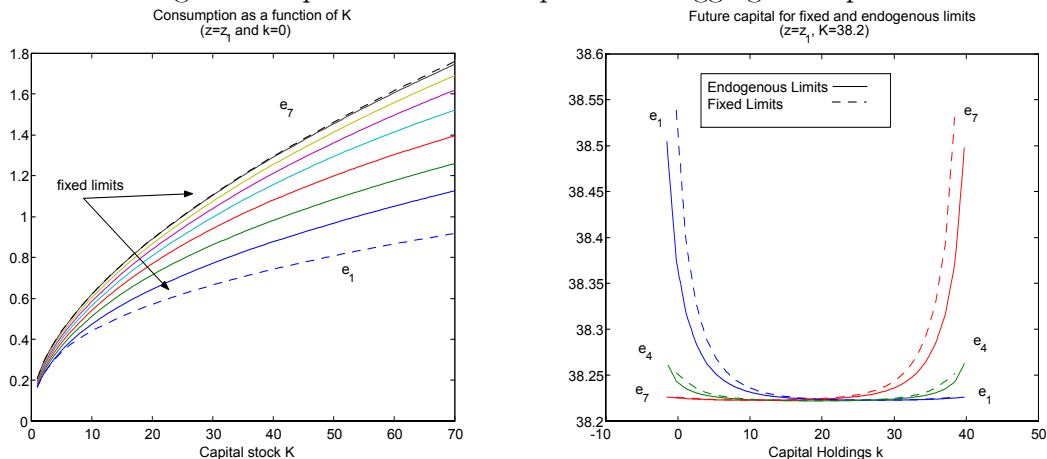
Suppose first that the aggregate capital stock is low. In this case, its relatively high return implies that every household wants to save (see Figure 6). In other words, a change in the labor income shock has a very little effect on the aggregate capital distribution, leading therefore to small G^ϵ terms and to very small CA effects. Note also that consumption in the trading arrangement is relatively low when capital is low, leading to a high marginal utility and to a very high and positive IM effect that outweighs the negative CA and CM effects. Thus, if capital is very low, we should expect a household with a higher income shock to have a lower incentive to default and therefore a looser limit. On the other hand, the opposite situation would arise with a high level of capital. In this case, consumption would be high and the positive IM effect would therefore be outweighed by the negative CM and CA effects. Clearly, this is a particularly appealing feature of the production economy with incomplete markets, since, at least for some stages of the transition to the stationary distribution, it can lead to the empirically more plausible case in which the borrowing constraints are looser for the high income agents. Finally, consider an intermediate level of aggregate capital inside the stationary distribution. As we mentioned earlier, capital accumulation is (partly) determined by the joint distribution of income and wealth. In particular, capital accumulation is mostly affected when low wealth coincides with low income. Therefore, when we move from very low income levels to intermediate ones, we expect to have a strong CA effect, leading to tighter limits. When low wealth is offset by a high idiosyncratic income, however, the CA effect is much lower, and the IM effect dominates. This explains the hump shape of the limits that we observe for intermediate levels of capital. These capital accumulation effects

are documented below on the right panel of Figure 5.

6.2. Comparison with Fixed Limits. As stated earlier, the model with incomplete markets and endogenous limits is compared to a model with exogenously fixed trading limits, which we set to zero as usually done in the existing literature. The policy functions for individual consumption and the law of motion of the aggregate capital stock are depicted below under fixed and endogenous limits. The left panel depicts individual consumption as a function of the aggregate capital stock for all levels of the idiosyncratic shock when type one households have zero wealth. As we see, individual consumption is increasing in aggregate capital because a higher aggregate capital leads to a higher wealth. Further, consumption inequality is higher under the tighter fixed zero limit, since households receiving a low income shock can borrow less and achieve less consumption smoothing in this case.

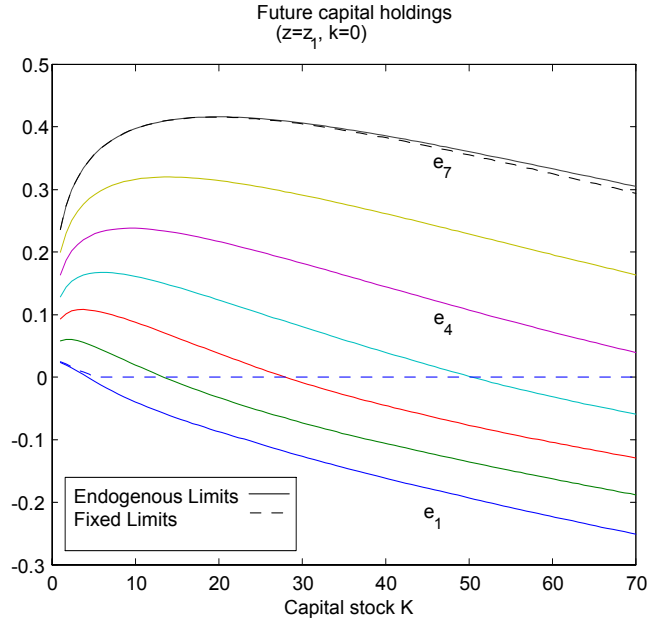
The right panel depicts the equilibrium law of motion of aggregate capital as a function of the wealth-income distribution. As we see, the figure reflects that a higher joint dispersion of individual capital holdings and income shocks ($k < 0$ and low ϵ) leads to a significantly higher capital accumulation under endogenous and fixed zero limits. As we mentioned above, when low income households are close to their borrowing constraint, market clearing requires that the return on capital is reduced, and this can be only done by increasing the next period's aggregate capital. As reflected by the figure, we observe the same effect but of a much smaller magnitude when the household has intermediate or high income shocks, confirming our previous statements on how aggregate capital accumulation depends on the joint distribution of income and wealth. Finally, we also see that the same pattern appears under fixed zero limits, in which case capital accumulation is higher due to the fact that households are likely to become constrained earlier and have therefore higher precautionary savings motives.

Figure 5: Equilibrium Consumption and Aggregate Capital



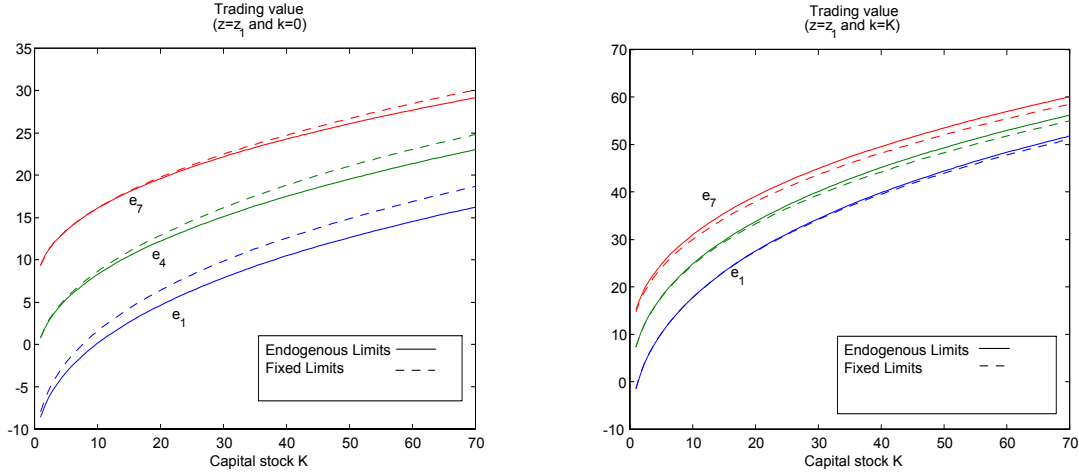
The following figure displays next period individual capital holdings under fixed and endogenous limits, where the key differences between the two set-ups become more apparent. As we see, the two models predict a very different behavior concerning the individual capital decisions for agents who enter the current period with zero wealth. Not surprisingly, agents with low labour income are borrowing constrained for most levels of aggregate capital. In contrast to this, and except for very low levels of the aggregate capital, where the return is extremely high, low income households are able to and willing to borrow in the endogenous limit case. Note that this is natural, since the borrowing constraint is looser than the one under fixed limits, and households can therefore borrow more. On the other hand, we also see that households borrow more in equilibrium when the aggregate capital stock increases. Clearly, this is due the fact that limits are endogenous and get looser as capital increases, while a higher aggregate level of capital also leads to lower expected interest payments on loans. Interestingly, we see that the level of savings for high income agents first increases with a higher aggregate capital then starts to fall. This is due to the combination of several effects. First, there is a precautionary savings motive both because of the incomplete markets structure and because of the presence of tight borrowing limits. This suggests that higher income agents, *ceteris paribus*, want to save more as capital increases. This effect is mitigated, however, for two reasons. First, the limits get looser with a higher aggregate capital, and the precautionary savings motive is therefore reduced. Second, the asset becomes a less attractive insurance device as its return becomes lower.

Figure 6: Equilibrium Individual Capital Holdings



The following figures explore the welfare effects of relaxing the limits from their fixed zero value to the level where they are not too tight.

Figure 7: Individual Welfare as a Function Aggregate Capital

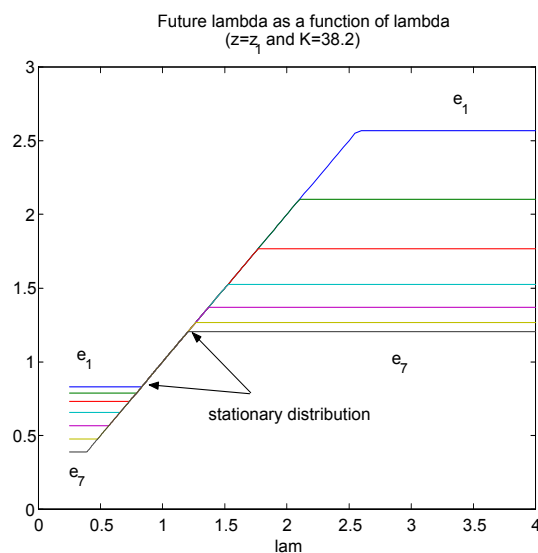


The left panel of Figure 7 depicts the value function $W(0, \epsilon; z, K)$ of a household with zero capital holdings, while the right panel depicts $W(K, \epsilon; z, K)$. By the symmetry of the idiosyncratic shock, this implies that the agents represented by the bottom lines on the left panel coexist with the agents represented by the top lines on the right panel. As we see, the value under fixed zero limits is higher for a household with zero capital holdings in spite of the fact that risk sharing is lower in this case. Further, the value of the symmetric household owning the entire capital stock and facing the opposite shock is lower under fixed zero limits. This means that relaxing the limits from the fixed no short-selling value to their endogenous value is not necessarily Pareto improving.

Whereas this seems to be counter-intuitive at first sight, note that it arises due to the general equilibrium effect discussed above, leading to a higher aggregate capital stock under fixed zero limits. This will benefit households who rely more on labour income, which is increasing in aggregate capital. In the above economy, these are the agents who have no capital holdings (left panel). Moreover we have seen that the lower their income the bigger is the general equilibrium effect, implying that lower income households gain more. On the other hand, a higher capital stock, leading to a lower capital return, hurts the high wealth households (right panel), since they undertake most of the investment. Clearly, these considerations can have interesting implications for the optimal way of setting the endogenous limits, an issue that we leave for future research.

6.3. Comparison with Complete Markets. This section compares the incomplete markets allocation with endogenous limits to the case in which markets are complete but households cannot commit on the trading contracts. In this case, we use as a state space the vector $(\lambda, \epsilon; z, K)$, where λ corresponds to the ratio of marginal utilities of types one and two. As mentioned above, λ measures the relative wealth of agent 2 compared to agent 1 in an indirect way. The following graph depicts the optimal law of motion for λ' as a function of the initial λ for a medium level of aggregate capital inside the stationary distribution and for different values of the idiosyncratic labor income shock.

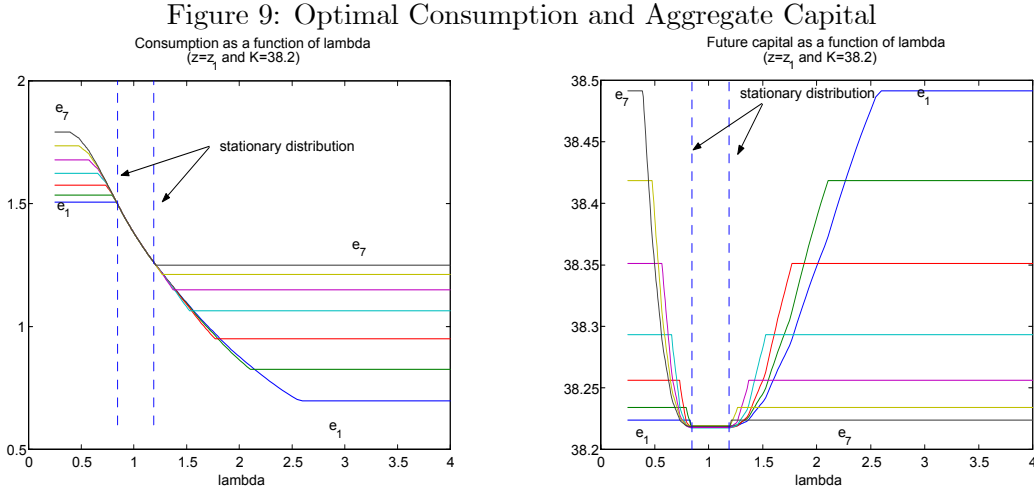
Figure 8: Future Relative Pareto Weight



The first thing reflected by the figure is that agents enjoy permanent perfect risk sharing in the long run. To see this, assume first that our initial λ is inside its stationary distribution $\lambda \in [0.83, 1.21]$. As we see on the graph, $\lambda' = \lambda$ inside this region, independently of the labor income shocks. First, condition (32) implies that this can only happen if neither agent's participation constraint is binding. Second, the same condition implies that the ratio of marginal utilities remains constant over time if this is the case. These facts, however, are the defining feature of a perfect risk sharing allocation. Assume now that we start with $\lambda > 2.5$, implying that agent 1 is entitled to receive less consumption than agent 2. In this case, Figure 8 implies that λ' depends on the idiosyncratic income of the agent, and that it will drop to a new level. In particular, the higher the idiosyncratic income the lower will be the new level of the relative Pareto weight, since type one agents require a higher compensation for staying in the risk sharing arrangement. This process will go on until the highest income

(ϵ^7) is experienced by the type 1 agents. In this case, λ will enter the stationary distribution ($\lambda = 1.21$) and remain constant forever, implying that agents enjoy permanent perfect risk sharing from that period on. In addition, a symmetric argument implies that whenever $\lambda < 0.83$, the relative Pareto weight will become 0.83 and remain constant forever after finite number of periods. Given this, the present framework implies that agents will obtain full insurance in the long-run, independently of the initial wealth distribution.

The policy functions for consumption and the law of motion of aggregate capital as a function of λ are depicted below. From the arguments above, it becomes clear that the allocations depend on the idiosyncratic income only outside the stationary distribution of λ . This is due to the fact that there is perfect consumption smoothing across individual income levels inside the stationary distribution of λ . The same is true for the aggregate capital accumulation. Interestingly, the more disperse is the wealth distribution (or the further away λ is from 1 outside of the stationary distribution) the higher is the aggregate capital accumulation, which is exactly the same effect of the wealth distribution that we detected in the incomplete market economies.



The previous results indicate that the complete markets solution of the present model is not very interesting. In particular, the introduction of capital accumulation makes default unattractive, resulting in a pure risk sharing allocation in the long run. Ábrahám and Cárceles-Poveda (2005) show that this result is not a consequence of the fact that we solved a suboptimal problem that does not internalize the effects of capital accumulation on the autarky utility. In addition, the authors show that a lower autarky penalty allowing agents to save in capital does not change the qualitative findings of the complete markets economy either. In sum, the incomplete markets framework is a more interesting case that results in empirically more plausible consumption patterns and borrowing limits.

7. CONCLUSIONS

The present work studies an economy with incomplete markets, capital accumulation and the possibility of default on financial liabilities. We show that the presence of a competitive intermediation sector that sets the borrowing limits implies that no symmetric equilibrium can allow for default. Among all these equilibria, we study the one with the loosest possible limits that prevent default. In particular, we characterize how the endogenous limits depend on the deep characteristics and on the state variables of the model.

For all levels of the income and technology shocks, we find that the endogenous limits become looser with a higher aggregate capital. Whereas a higher capital stock increases both the trading and autarky values through its effect on wages, it reduces the capital return and therefore the financial liabilities, leading to a lower incentive to default. In addition, we also find that limits are non-monotonic in the idiosyncratic labor income, whereas default incentives may decrease with income. These last results are due to the presence of both incomplete markets and capital accumulation, implying that a higher labor income affects both the trading and autarky values, considerably altering the incentives to default. Further, these results are in sharp contrast with the findings of the complete markets literature with enforcement constraints, where a higher income only affects the autarky value and unambiguously increases the incentives to default.

The allocations with endogenous limits are also compared to the ones resulting from fixed zero limits and from a setup with complete markets and capital accumulation. Regarding the first comparison, we find that the endogenous limits that prevent default in equilibrium can be significantly different from zero, leading to a lower consumption inequality and to a lower capital accumulation than under fixed zero limits. Further, since the limits become looser with a higher aggregate capital, households tend to borrow more when capital increases when the limits are endogenous, whereas they tend to save under fixed limits. Finally, the higher capital accumulation under fixed zero limits implies that loosening the borrowing constraints from zero to their endogenous value is not necessarily Pareto improving.

As to the comparison with a complete markets environment, we find that the computation of our equilibrium is considerably more complicated when markets are incomplete, since no obvious central planning problem yields the same allocation. On the other hand, a context with complete markets and capital accumulation is not very interesting, since it implies a perfect risk sharing allocation. This result is clearly at odds with empirical data and it provides a strong motivation for moving towards the incomplete market economies.

In addition, we believe that our findings can be applied to a variety of other interesting contexts with incomplete markets and endogenous limits. As an example, Krueger and Perri (2003) analyze consumption and wealth inequality in a context with complete markets and enforcement constraints, providing a possible explanation as of why the increase in earnings

inequality has not been accompanied by an increase in consumption inequality in recent decades. As argued by the authors, borrowing limits might have become looser due to a change in the exogenous earning process, whereas our results suggest that growth (capital accumulation) could have the same effect. In addition, Aiyagari et al (2002) study optimal government debt and fiscal policy in a context where the government wants to accumulate assets to smooth taxes and insure against future expenditure shocks. This also implies that households will be in debt in the long-run. On the other hand, the possibility of default reflected in the endogenous borrowing limits could impose constraints on how much assets the government can accumulate. More generally, our results suggests that fiscal policy and social insurance programs can have significant effects on the level of the endogenous trading constraints, and the welfare analysis of any policy reform should therefore take this into account. This is particularly relevant for the study of social security reforms, where the level of the endogenous limits could considerably affect the financial viability and impact of the reform. These are important issues that we leave for further research.

APPENDIX

Appendix 1. The trading and autarky values can be expressed as:

$$W(s_t) = u(g^c(s_t)) + \beta \int_{\varepsilon_\epsilon} \int_{\varepsilon_z} W(s_{t+1}) d\varepsilon_\epsilon d\varepsilon_z$$

$$V(s_t) = u(w(K, z)\epsilon) + \beta \int_{\varepsilon_\epsilon} \int_{\varepsilon_z} V(s_{t+1}) d\varepsilon_\epsilon d\varepsilon_z$$

Differentiating the autarky value with respect to $j \in \{k, \epsilon, z, K\}$, the derivatives with respect to the state variables can be expressed as an infinite sum as follows:

$$V_k(s_t) = E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} \epsilon_\tau u'(c_\tau^{au}) \frac{\partial w(K_\tau, z_\tau)}{\partial K} G^k(s_{\tau-1})$$

$$V_\epsilon(s_t) = E_t \sum_{\tau=t}^{\infty} (\beta \rho_\epsilon)^{\tau-t} w(K_\tau, z_\tau) u'(c_\tau^{au}) + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} \frac{\partial w(K_\tau, z_\tau)}{\partial K} u'(c_\tau^{au}) \epsilon_\tau G^\epsilon(s_{\tau-1})$$

$$V_z(s_t) = E_t \sum_{\tau=t}^{\infty} (\beta \rho_z)^{\tau-t} \epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial z} u'(c_\tau^{au}) + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} \epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K} u'(c_\tau^{au}) G^z(s_{\tau-1})$$

$$V_K(s_t) = \epsilon_t \frac{\partial w(K_t, z_t)}{\partial K} u'(c_t^{au}) + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} \epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K} u'(c_\tau^{au}) G^K(s_{\tau-1})$$

where $c_\tau^{au} = \epsilon_\tau w(K_\tau, z_\tau)$ represents the consumption in autarky. To define $G^k(s_{\tau-1})$, we let Γ_τ^k be a set of sequences of indices of k or K with length $\tau - t + 1$ for every $\tau > t$. The first

element of the sequence is always k , since we differentiate with respect to k , and the last one is K , since the long-lasting effects of a change in k are through the effect of K on wages. The set of all possible sequences for a given τ can then be defined as:

$$\Gamma_\tau^k = \{i_0, i_1, i_2, \dots, i_{\tau-t} : i_j \in \{k, K\} \text{ for } \forall j: 0 < j < \tau - t, i_0 = k, i_{\tau-t} = K\}.$$

Given this, it follows that,

$$G^k(s_{\tau-1}) = \sum_{\mathbf{i} \in \Gamma_\tau^k} g_{i_0}^{i_1}(s_t) g_{i_1}^{i_2}(s_{t+1}) \dots g_{i_{\tau-t-1}}^{i_{\tau-t}}(s_{\tau-1}).$$

To define $G^\epsilon(s_{\tau-1})$, we let $\Gamma_\tau^{\epsilon^i}$ be the set of sequences of indices of k , K and ϵ with length $\tau - t + 1$ for every $\tau > t$, such that the first element of the sequence is ϵ , since we differentiate with respect to ϵ , and the last one is K , since the long-lasting indirect effects are through the effect of K on wages. Furthermore, no ϵ^i in the sequence can follow any k or K . The set of all possible sequences for a given τ can then be defined as:

$$\begin{aligned} \Gamma_\tau^{\epsilon^i} &= \{i_0, i_1, i_2, \dots, i_{\tau-t} : i_j \in \{k, K, \epsilon^i\} \text{ for } \forall j: 0 < j < \tau - t, \\ i_0 &= \epsilon^i, i_{\tau-t} = K \text{ and if } i_j \neq \epsilon^i \text{ then } \forall l > j \ i_l \neq \epsilon^i\}. \end{aligned}$$

Given this, we have that,

$$G^\epsilon(s_{\tau-1}) = \sum_{\mathbf{i} \in \Gamma_\tau^{\epsilon^i}} g_{i_0}^{i_1}(s_t) g_{i_1}^{i_2}(s_{t+1}) \dots g_{i_{\tau-t-1}}^{i_{\tau-t}}(s_{\tau-1}).$$

Notice that the first l elements in the above product can be replaced by ρ_ϵ^l whenever $i_0 = i_1 = \dots = i_l = \epsilon^i$. $G^z(s_{\tau-1})$ can be defined similarly. In particular, we let Γ_τ^z be the set of sequences of indices of k , K and z with length $\tau - t + 1$ for every $\tau > t$ such that the first element of the sequence is z , since we differentiate with respect to z , and the last one is K , since the long-lasting indirect effects are through the effect of K on wages. Further, no z in the sequence can follow any k or K . The set of all possible sequences for a given τ can then be defined as:

$$\begin{aligned} \Gamma_\tau^z &= \{i_0, i_1, i_2, \dots, i_{\tau-t} : i_j \in \{k, K, z\} \text{ for } \forall j: 0 < j < \tau - t, \\ i_0 &= z, i_{\tau-t} = K \text{ and if } i_j \neq z \text{ then } \forall l > j \ i_l \neq z\}. \end{aligned}$$

Given this, we have that,

$$G^z(s_{\tau-1}) = \sum_{\mathbf{i} \in \Gamma_\tau^z} g_{i_0}^{i_1}(s_t) g_{i_1}^{i_2}(s_{t+1}) \dots g_{i_{\tau-t-1}}^{i_{\tau-t}}(s_{\tau-1}).$$

Notice that the first l elements in the above product can be replaced by ρ_z^l whenever $i_0 = i_1 = \dots = i_l = z$.

Finally, to define $G^K(s_{\tau-1})$, we let Γ_τ^K be the set of sequences of indices of k and K with length $\tau - t + 1$ for every $\tau > t$ such that the first element is K , since we differentiate with respect to K , and the last one is K , since all these long-lasting effects are through the effect of K on wages. The set of all possible sequences for a given τ can then be defined as:

$$\Gamma_\tau^K = \{i_0, i_1, i_2, \dots, i_{\tau-t} : i_j \in \{k, K\} \text{ for } \forall j: 0 < j < \tau - t, i_0 = K, i_{\tau-t} = K\}.$$

Given this, we have that:

$$G^K(s_{\tau-1}) = \sum_{\mathbf{i} \in \Gamma_\tau^K} g_{i_0}^{i_1}(s_t) g_{i_1}^{i_2}(s_{t+1}) \dots g_{i_{\tau-t-1}}^{i_{\tau-t}}(s_{\tau-1}).$$

As to the signs of the previous terms, consider first G^k . As shown above, the sign of G^k crucially depends on the sign of the policy function derivatives g_k^K , g_K^K , g_k^k and g_K^k . In the present framework, $g_K^K > 0$ due to the fact that a higher initial aggregate capital leads to a higher aggregate investment. Second, $g_k^K < 0$ for low levels of k due to the fact that a higher k (or a lower wealth dispersion) leads to a lower capital accumulation. As explained in the main text, a higher wealth dispersion implies that low wealth households are closer to their borrowing limit. Given this, the return on capital has to decrease so that high wealth households do not want to save so much and market clearing is achieved. In this case, this can only be achieved by increasing capital accumulation. Conversely, a higher k (or a lower wealth dispersion) leads to a lower capital accumulation, and this implies that $g_k^K < 0$. Third, $g_k^k > 0$ due to the fact that a higher level of initial asset holdings increases the individual wealth and leads to a higher individual investment. Finally, our numerical results suggest that $g_K^k > 0$ for very low levels of the aggregate capital stock and $g_K^k < 0$ otherwise. Using the definition of G^k , these results suggest that $G^k < 0$. Finally, we can use similar arguments to infer that G^K and G^z are expected to be positive and G^ϵ is expected to be negative.

Differentiating the trading value with respect to $j \in \{k, \epsilon, z, K\}$, the derivatives of the autarky value with respect to these variables can be expressed as an infinite sum as follows:

$$\begin{aligned} W_K(s_t) &= u'(c_\tau) \left[\epsilon_t \frac{\partial w(K_t, z_t)}{\partial K} + k_t \frac{\partial r(K_t, z_t)}{\partial K} \right] + \\ &+ E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau) \left[\epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K} \right] G^K(s_{\tau-1}) \\ &+ E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau) \left[k_\tau \frac{\partial r(K_\tau, z_\tau)}{\partial K} \right] G^K(s_{\tau-1}) \end{aligned}$$

$$\begin{aligned}
W_k(s_t) &= u'(s_t)(1 + r(K, z) - \delta) + \\
&\quad + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau) \epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K} G^k(s_{\tau-1}) \\
&\quad + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau) k_\tau \frac{\partial r(K_\tau, z_\tau)}{\partial K} G^k(s_{\tau-1}) \\
W_\epsilon(s_t) &= E_t \sum_{\tau=t}^{\infty} (\beta \rho_\epsilon)^{\tau-t} u'(c_t) w(K, z) + \\
&\quad + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau) \left[\epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K} + k_\tau \frac{\partial r(K_\tau, z_\tau)}{\partial K} \right] G^\epsilon(s_{\tau-1}) \\
W_z(s_t) &= E_t \sum_{\tau=t}^{\infty} \left((\beta \rho_z)^{\tau-t} u'(c_t) \left[\epsilon_\tau \frac{\partial w(K, z)}{\partial z} + k_t \frac{\partial r(K, z)}{\partial z} \right] \right) \\
&\quad + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u'(c_\tau) \left[\epsilon_\tau \frac{\partial w(K_\tau, z_\tau)}{\partial K} + k_\tau \frac{\partial r(K_\tau, z_\tau)}{\partial K} \right] G^z(s_{\tau-1})
\end{aligned}$$

where $c_\tau = \epsilon_\tau w(K_\tau, z_\tau) + (r(K_\tau, z_\tau) + 1 - \delta)k_\tau - k_{\tau+1}$ represents the consumption in the trading contract and the terms G^j for $j \in \{k, \epsilon, z, K\}$ are defined as above.

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